# The plastic 'inclusion' as a bridge between local crack plasticity and the global elastic field

M. N. James
*Faculty of Science and Technology, University of Plymouth, Plymouth (UK)*
*Department of Mechanical Engineering, Nelson Mandela Metropolitan University (South Africa)*
*mjames@plymouth.ac.uk*

C. J. Christopher, Yanwei Lu
*Faculty of Science and Technology, University of Plymouth, Plymouth (UK)*

E. A. Patterson
*School of Engineering, University of Liverpool, Liverpool, (UK)*
*Eann.Patterson@liverpool.ac.uk*

**ABSTRACT.** This paper presents a novel 'plastic inclusion' concept for dealing with the local plasticity which occurs at the tip of a growing fatigue crack. Such localised plasticity essentially blunts the crack and induces stresses which act on the applied elastic stress field at the boundary of the elastic-plastic enclave surrounding the crack. The paper outlines a model of crack tip stresses that explicitly incorporates these stresses and also those that might arise from crack wake contact. This offers a way of reconciling local crack tip plasticity-induced stress field perturbations with the driving force of the overall elastic stress field. The outcome is the identification of a modified crack tip stress intensity factor which should be able to explicitly predict the magnitude of plasticity-induced crack tip shielding and resolve the many controversies associated with plasticity-induced shielding. A full-field approach is developed for stress using photoelasticity and also for displacement using digital image correlation.

**KEYWORDS.** Plastic enclave; Crack tip stress; Plastic notch; Crack shielding; Muskhelishvili; T-stress.

## INTRODUCTION

The distinction between a notch and a crack is a function of their relative acuity, which is affected by, and subsumed within, the consequences of the development of a plastic enclave around the entire crack. In ductile structural materials experiencing crack growth by fatigue, it would therefore seem eminently sensible to consider the crack as a 'plastic inclusion', or notch, in the material and to consider the stress components in the *x-y* plane that arise from its containment within a surrounding elastic body.

This is the approach adopted by the present authors arising from the insights gained as a result of some 30 years of research into fatigue crack closure and experimental mechanics. The power of experimental mechanics is that certain techniques are capable of providing full field stress information for cracked bodies. If an appropriate mathematical description is available for a crack surrounded by a plastic enclave, then detailed comparisons between theory and experiment can be done on a full field basis. Such an approach leads to the identification of a modified stress intensity factor perpendicular to the crack, $K_F$, which drives the crack forwards and which explicitly includes the influences of plasticity-induced shielding on the global applied elastic field. The analysis also provides a stress intensity parameter acting parallel to the crack that acts to retard growth, $K_R$, and a compatibility-induced shear stress intensity factor, $K_S$ [1]. This new model has been experimentally validated against full field crack tip stress data obtained from photoelasticity [2] and also, to a more limited extent, by digital image correlation and appears to offer useful insights into a more generalised way of dealing with plasticity-induced shielding in fatigue crack growth. The observed trends in the three new stress intensity factors as a function of increase in crack length and change in stress ratio appear to be physically meaningful and self-consistent.

This paper therefore presents an outline of the development, verification and application of this new model for crack tip stress fields in the presence of a plastic enclave around a growing fatigue crack. The model is independent of the mechanisms of plastic deformation and therefore potentially applicable to a variety of materials. It can also be used to mathematically explore the effect on shape of the crack tip stress field of changes in magnitude of the various parameters; for, example, the effect of variation in magnitude of increasing positive or negative *T*-stress on crack tip fringe patterns representing difference between the principal stresses. Such fringes are directly comparable with photoelastic fringe patterns and the model supports the conclusions reached by Roychowdhury and Dodds [3] and Kim et al [4] regarding the influences of *T*-stress. Possible extensions to the current model will be considered in the journal paper to be submitted to the *International Journal of Fatigue*.

## THEORETICAL BACKGROUND

The concept of a plastic 'inclusion' was originally proposed by Eshelby [5] as a useful mathematical approach to dealing with 'composite' materials, i.e. those where part of the material has undergone an 'instantaneous' change in properties, such as in a phase transformed zone or a region of plasticity. This concept offers the possibility of interpreting the influences on the applied elastic stress field of the plastic enclave that is generated during crack growth. The mathematics of this approach are, however, quite intensive and, as a first step in assessing the viability of such an approach, the present authors have developed a crack tip stress model that incorporates realistic assumed influences of the plastic enclave.

The concept rests on capturing the effects of the crack tip and crack wake plasticity through elastic stress distributions applied at the elastic-plastic boundary, which allows an elastic model to adequately capture the influences of plasticity without needing the involvement of plastic deformation mechanisms. This affords a significant simplification in the mathematical complexity. The stresses considered include a possible wake contact stress, which decays behind the crack tip as a power law function with power = -½, and stresses induced by compatibility requirements between the elastic material with a Poisson's ratio of perhaps 0.3 and the plastic region which undergoes constant volume deformation ($\nu$ = 0.5). A Muskhelishvili complex potential extension to the Williams crack tip stress field is found for four stress parameters representing a stress acting perpendicular to the crack plane, a *T*-stress acting parallel to the crack plane, a wake contact stress acting perpendicular to the crack plane and a compatibility-induced shear stress parallel with the crack plane acting at the elastic-plastic boundary. This model has been validated via full field fitting to photoelastic stress fringe patterns, obtained from epoxy resin and polycarbonate specimens. It has also been extended to the displacement fields measured in digital image correlation techniques, which allows its application to metallic alloys.

### *Four-Parameter Model for Stress (Photoelasticity)*

The development of this model is described in references 1 and 2. The stresses included in the model are repeated in the free body diagram shown in Fig. 1, as this diagram is fundamental to understanding the concepts underlying the model. The forces acting at the interface of the plastic enclave and elastic field near the crack tip include: $F_{Ax}$ and $F_{Ay}$, the reaction to the remote load that generates the crack tip stress fields traditionally characterised by the stress intensity factor $K_1$; $F_T$, due to the *T*-stress which depends on the remote load for a specific configuration; $F_{Px}$ and $F_{Py}$, induced by the compatibility requirements on the elastic plastic boundary near the crack tip, resulting from the permanent deformation in the plastic zone around the crack tip, which is extensive in the direction perpendicular to the crack and contractive along the crack due to the effect of the Poisson's ratio; $F_S$, induced by the compatibility requirements on the elastic-plastic boundary of the crack wake, as plastic deformation is a constant volume effect, i.e. $\nu$ = 0.5, while elastic deformation occurs with $\nu$ = 0.3, $F_S$ could change its direction when the crack becomes open from closed state; $F_C$, is the contact force arising from the plastic wake contact effect transmitted to the elastic-plastic boundary of the crack wake.

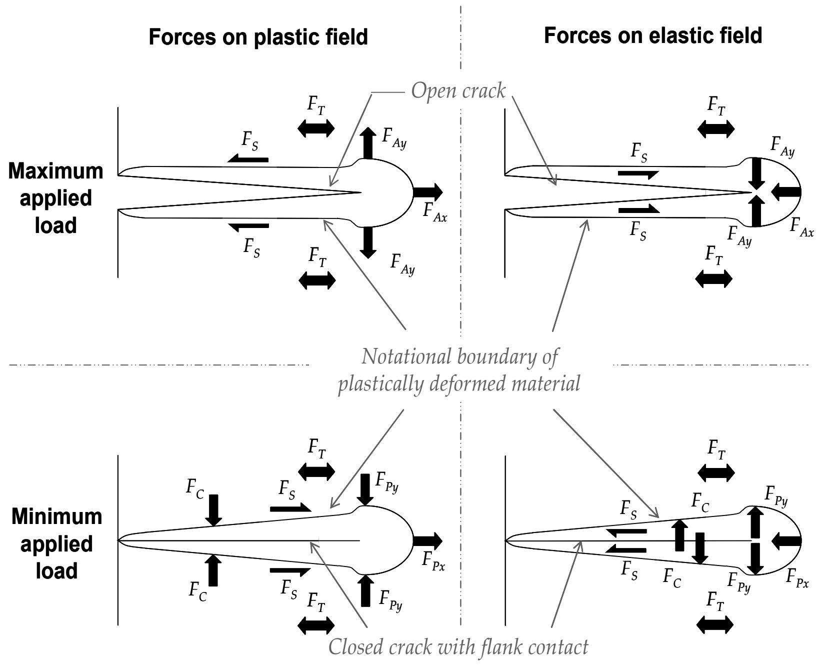


Figure 1: Schematic idealisations of forces acting at the interface of the plastic enclave and the surrounding elastic material, where $F_A$ is the applied force generating the crack tip stress field characterized by $K_I$, $F_T$ represents the force due to the T-stress shown in this example as positive, $F_S$ is the interfacial shear force between the elastic and plastic zones, $F_C$ and $F_P$ together create the shielding effect. $F_P$ is the force generated by the constraint of compatibility on the plastically deformed material and $F_C$ is the contact force between the flanks of the crack generated by the interference of the plastic zones along the flanks. After reference 1.

In transmission photoelasticity, the relationship between stress fields and the isochromatic fringe data is given by:

$$\frac{Nf}{h} = \left(\sigma_1 - \sigma_2\right) = \sqrt{\left(\sigma_y - \sigma_x\right)^2 + \left(2\sigma_{xy}\right)^2} \tag{1}$$

where, $\sigma_1, \sigma_2$ are principal stresses; $\sigma_x$ , $\sigma_y$ are stresses in $x$ and $y$ direction respectively; $\sigma_{xy}$ represents shear stress; $N$ is fringe order; $h$ is specimen thickness and $f$ is material fringe constant.
Using Muskhelishvili's potential functions, Eq. (1) can be rewritten as:

$$\frac{Nf}{h} = \left(\sigma_1 - \sigma_2\right) = \left|\sigma_y - \sigma_x + 2i\sigma_{xy}\right| \tag{2}$$

Hence,

$$\frac{Nf}{h} = 2\left|\bar{z}\varphi''(z) + \chi''(z)\right| = 2\left|\bar{z}\,\phi'(z) + \psi(z)\right| \tag{3}$$

where, $\varphi(z)$ and $\chi(z)$ are the two analytical functions in Muskhelishvili's approach, and $\phi(z)$ and $\psi(z)$ are known as Muskhelishvili's potential functions. $\phi(z) = \varphi'(z)$ , and $\psi(z) = \chi''(z)$ .
Williams' solution for stress fields around the crack tip can be expressed in terms of Cartesian co-ordinates under mode I loading as:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} cos\left(\frac{\theta}{2}\right)\left[1 - sin\left(\frac{\theta}{2}\right) sin\left(\frac{3\theta}{2}\right)\right] + T + O\left(r^{\frac{1}{2}}\right) \tag{4a}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} cos\left(\frac{\theta}{2}\right)\left[1 + sin\left(\frac{\theta}{2}\right) sin\left(\frac{3\theta}{2}\right)\right] + O\left(r^{1/2}\right) \tag{4b}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} sin\left(\frac{\theta}{2}\right) cos\left(\frac{\theta}{2}\right) cos\left(\frac{3\theta}{2}\right) + O\left(r^{1/2}\right) \tag{4c}$$

where $r$ and $\theta$ are the distance and polar angle measured from the crack tip and crack plane respectively. Substituting the leading terms in Williams' solution into Eq. (2), the following equation can be obtained:

$$\frac{Nf}{h} = \left|\sigma_y - \sigma_x + 2i\sigma_{xy}\right| = \left|Az^{-1/2} + Bz^{-3/2}\,\overline{z} + Cz^0\right| \tag{5}$$

where

$$A = -B = \frac{K_I}{2\sqrt{2\pi}},\quad A + B = 0\ ,and\quad C = -T$$

The new four-parameter mathematical model was obtained by adding two new terms into Eq. (5):

$$\left|\sigma_y - \sigma_x + 2i\sigma_{xy}\right| = \left|Az^{-1/2} + Bz^{-3/2}\,\overline{z} + Cz^0 + Dz^{-1/2}ln\left(z\right) + Ez^{-3/2}\,\overline{z}ln\left(z\right)\right|$$

$$\frac{Nf}{h} = \left|Az^{-1/2} + Bz^{-3/2}\,\overline{z} + Cz^0 + Dz^{-1/2}ln\left(z\right) + Ez^{-3/2}\,\overline{z}ln\left(z\right)\right| \tag{6}$$

where, $z$ is the complex co-ordinate in the physical plane, $z = x + iy$; $x\,and\,y$ are co-ordinates in a Cartesian system with the origin at the crack tip; and $A,B,C,D,E$ are unknown coefficients that need to be determined. In this model, an assumption has been made by taking $D + E = 0$ in the mathematical analysis in order to give an appropriate asymptotic behaviour of the stress along the crack flank, and $A + B \neq 0$ if an interfacial shear stress exists at the interface of the elastic plastic boundary (if not, $A + B = 0$).

This mathematical model describes the elastic stress field around the tip of fatigue cracks which are acting as a blunt plastic enclave in the material subject to Mode I loading. The terms $Dz^{-1/2}ln\left(z\right)$ and $Ez^{-3/2}\,\overline{z}ln\left(z\right)$ were used to include the shielding effect of the plastic enclave surrounding the fatigue crack on the elastic fields around the crack tip. These are characterized via two parameters, i.e. a crack retardation stress intensity and an interfacial shear stress intensity. Thus, the stress field in the elastic material can be characterized by four parameters: a stress intensity factor analogous to $K_1$, called $K_F$ in our paper, the $T$-stress, an interfacial shear stress intensity factor, $K_S$, and a retarding stress intensity factor, $K_R$. The elastic stress fields near the crack tip are then given by:

$$\sigma_x = -\frac{1}{2}\left(A + 4B + 8E\right)r^{-\frac{1}{2}}cos\frac{\theta}{2} - \frac{1}{2}Br^{-\frac{1}{2}}cos\frac{5\theta}{2} - C - \frac{1}{2}Er^{-\frac{1}{2}}\left[ln\left(r\right)\left(cos\frac{5\theta}{2} + 3cos\frac{\theta}{2}\right) + \theta\left(sin\frac{5\theta}{2} + 3sin\frac{\theta}{2}\right)\right] + \mathcal{O}\left(r^{\frac{1}{2}}\right)$$

$$\sigma_y = \frac{1}{2}\left(A - 4B - 8E\right)r^{-\frac{1}{2}}cos\frac{\theta}{2} + \frac{1}{2}Br^{-\frac{1}{2}}cos\frac{5\theta}{2} + \frac{1}{2}Er^{-\frac{1}{2}}\left[ln\left(r\right)\left(cos\frac{5\theta}{2} - 5cos\frac{\theta}{2}\right) + \theta\left(sin\frac{5\theta}{2} - 5sin\frac{\theta}{2}\right)\right] + \mathcal{O}\left(r^{\frac{1}{2}}\right) \tag{7}$$

$$\sigma_{xy} = -\frac{1}{2}r^{-\frac{1}{2}}\left[Asin\frac{\theta}{2} + Bsin\frac{5\theta}{2}\right] - Er^{-\frac{1}{2}}sin\theta\left[ln\left(r\right)cos\frac{3\theta}{2} + \theta sin\frac{3\theta}{2}\right] + \mathcal{O}\left(r^{\frac{1}{2}}\right)$$

Four parameters are therefore used to characterize the stress fields generated by the forces in Fig. 1 an opening mode stress intensity factor $K_F$, the shear stress intensity factor $K_S$, the retardation stress intensity factor , and the $T$-stress. $K_F$ is defined from the asymptotic limit of $\sigma_y$ as $x \rightarrow +0$, along $y = 0$, i.e. towards the crack tip from the front along the crack line:

$$K_F = \lim_{r \to 0}\left[\sqrt{2\pi r}\left(\sigma_y + 2Er^{-1/2}ln(r)\right)\right] = \sqrt{\frac{\pi}{2}}\left(A - 3B - 8E\right) \quad (8)$$

The quantity $K_R$ characterises the direct stresses acting parallel to the crack growth direction and was obtained by evaluating $\sigma_x$ in the limit as $x \to -0$, along $y = 0$, i.e. towards the crack tip from behind along the crack flank:

$$K_R = \lim_{r \to 0}\left[\sqrt{2\pi r}\sigma_x\right] = \frac{\pi^{3/2}}{\sqrt{2}}\left(D - 3E\right) \quad (9)$$

The quantity $K_S$ characterises the shear stress on the elastic-plastic boundary, and is derived from the asymptotic limit of $\sigma_{xy}$ as $x \to -0$, along $y = 0$, i.e. towards the crack tip from behind along the crack flank:

$$K_S = \lim_{r \to 0}\left[\sqrt{2\pi r}\sigma_{xy}\right] = \mp\sqrt{\frac{\pi}{2}}\left(A + B\right) \quad (10)$$

A +ve sign indicates $y > 0$, and a -ve sign that $y < 0$.

*T*-stress is the transverse stress which is added to $\sigma_x$ as a constant term and is given by

$$T = -C$$

This new four-parameter model, as expressed in Eq. (6), has been termed the CJP (Christopher-James-Patterson) model by the authors, and describes the 2D elastic stress fields near the crack tip for a crack which is essentially acting as a plastic notch in the material and relates the stress fields to the value of isochromatic fringe order N in photoelastic images. One advantage of using this plastic inclusion model for the stress field at the tip of a fatigue crack, is that it provides insight into one of the fundamental questions associated with cracking, that of why and when a crack and a notch can be considered to be similar in behaviour. It also attempts to answer, at least to some extent, the fundamental question of how fatigue crack growth, a phenomenon which explicitly derives from plastic deformation, can be accurately described by an elastically-derived parameter, i.e. the stress intensity factor.

### *Four-Parameter Model for Displacement (Digital Image Correlation)*

Photoelasticity is rather limited in application to metallic materials and although this model works well in describing fatigue crack growth in polycarbonate, it was desired to extend it to describing crack growth in metallic materials. This is possible using digital image correlation (DIC) techniques, which can measure displacement on the surface of cracked specimens. The experimental application of both photoelastic and DIC techniques to crack tip stress characterisation has been described in reference [6]. The new four parameter model can be solved from Muskhelishvili's stress potential functions for direct application to displacement fields as given below:

$$2\mu\left(u_x + iu_y\right) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\chi'(z)}$$

$$\begin{aligned}2\mu\left(u_x + iu_y\right) = {} & \kappa\left(-2\left(B+2E\right)z^{\frac{1}{2}} + 4Ez^{\frac{1}{2}} - 2Ez^{\frac{1}{2}}ln(z) - \frac{C-F}{4}z\right) \\ & - z\left(-\left(B+2E\right)\bar{z}^{-\frac{1}{2}} - E\bar{z}^{-\frac{1}{2}}\overline{ln(z)} - \frac{C-F}{4}\right) \\ & - \left(A\bar{z}^{\frac{1}{2}} + D\bar{z}^{\frac{1}{2}}\overline{ln(z)} - 2D\bar{z}^{\frac{1}{2}} + \frac{C+F}{2}\bar{z}\right)\end{aligned} \quad (12)$$

where, $u_x$ and $u_y$ represent horizontal and vertical displacement respectively, $\mu = \frac{E}{2(1+\upsilon)}$; $\kappa = 3 - 4\nu$ (Plane strain) or $\kappa = \frac{3-\nu}{1+\nu}$ (Plane stress).

The stress intensity factor driving crack growth, $K_F$, the interfacial stress intensity factor $K_S$, the retardation stress intensity factor $K_R$ and the *T*-stress (which has to be found as components $T_x$ in x-direction and $T_y$ in y-direction) can be calculated using Eqs. (8-11) respectively with the small change that:

$$T_X = -C \qquad (13)$$
$$T_y = -F$$

## EXPERIMENTAL VERIFICATION USING POLYCARBONATE

Some work has already been reported in the literature that was aimed at validating the existence of the compatibility-induced stresses that underlie the new stress intensity factors defined in the CJP model [1, 2]. The work to be presented at this conference has sought to explore issues such as the repeatability of data between duplicate tests, and to determine whether the observed trends in the parameters as a function of crack length and stress ratio are sensible. It has also examined how best to optimise the fit between the experimental photoelastic and theoretical data.
Fig. 2 and Table 1 demonstrate the increased accuracy of fit to full field experimental photoelastic data arising from use of the new four-parameter CJP model compared with the standard Williams solution for crack tip stresses. Better values are obtained for all three fit parameters, implying that the CJP model captures the subtleties of local plastic perturbation of the global elastic stress field much better than is possible with the purely elastic Williams model.

| | Williams' model | | | | CJP model | | | |
|---|---|---|---|---|---|---|---|---|
| | 2 parameters | | 4 parameters | | 4 parameters | | | |
| | $K_I$ | $T$ | $K_I$ | $T$ | $K_F$ | $K_S$ | $K_R$ | $T$ |
| | 1.563 | 3.649 | 1.585 | 3.976 | 1.381 | -0.321 | 0.202 | 3.913 |
| *mean* | $\mu$ = 0.0019 | | $\mu$ = 0.0124 | | $\mu$ = -0.0074 | | | |
| *Std Dev* | $s$ = 0.0418 | | $s$ = 0.0371 | | $s$ = 0.0264 | | | |
| *residual* | $\varrho$ = 0.0050 | | $\varrho$ = 0.0044 | | $\varrho$ = 0.0025 | | | |

Table 1: Comparison between output from the Williams and CJP crack tip stress models and experimental photoelastic data. All three fit parameters give better values with the new model.

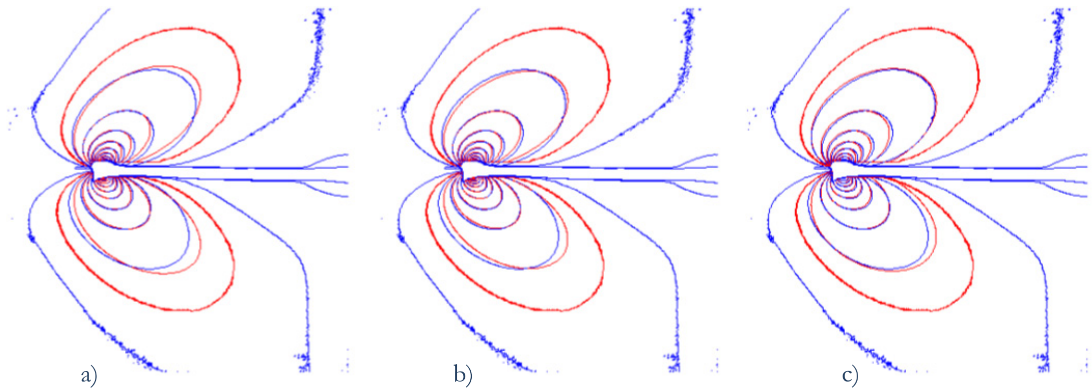


Figure 2: Comparison between experimental fringe patterns (blue) and theoretical solutions (red) for: (a) the 2-term Williams model; (b) the 4-term Williams model; (c) the new CJP model with 4 parameters which incorporates two terms arising from plasticity-induced stresses taken as acting at the elastic-plastic boundary.

## CONCLUSIONS

This paper has outlined the thinking and developmental process behind this innovative model for a multi-parameter characterisation of the stress field around a crack contained within a plastic enclave. Realistic elastic stress distributions representing crack wake contact and the effects of compatibility acting at the elastic-plastic boundary have been modelled, treating the crack as plastic inclusion in an elastic body. The model has led to the definition of a modified stress intensity factor perpendicular to the crack plane, called $K_F$, which drives crack growth in an analogous fashion to $K_I$. The shielding effect of the plastic enclave is accounted for via two new stress intensity factors, an interfacial shear stress intensity factor, $K_S$, and a retarding stress intensity factor, $K_R$. The forces leading to these stress intensity factors have been schematically represented using a free body diagram (Fig. 1) and the model also provides a value for the *T*-stress. Output from the model has been verified via full field comparison with experimental photoelastic stress analysis and the equations have been solved for the case of displacement, which then allows the use of experimental full field digital image correlation (DIC) data.

## REFERENCES

[1] C. J. Christopher, M. N. James, E. A. Patterson, K. F. Tee, Int. J. Fract., 148 (2007) 361.
[2] C. J. Christopher, M. N. James, E. A. Patterson, K. F. Tee, Engng Fract. Mech., 75 (2008) 4190.
[3] S. Roychowdhury, R. H. Dodds, Int. J. Solids Structs, 41 (2004) 2581.
[4] Y. Kim, X. K. Zhu, Y. J. Chao, Engng Fract. Mech., 68 (2001) 895.
[5] W. S. Slaughter, The Linearized Theory of Elasticity, Section 9.4, Birkhausen, Boston (2001).
[6] E. A. Patterson, M. N. James, In: Proc. Int. Conf. on Crack Tip Stress Characterisation, Forni di Sopra, Udine, Italy (2011)