A unified stress-based approach to assess failure in solids and liquids

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ABSTRACT. Failure in continuum media, either solids or liquids, can be regarded as a physical phenomenon which mathematically can be identified by the fulfilment of a limit condition usually involving the stress state inside the material. By considering classical failure theories for solids (plasticity and fracture mechanics approach) and a recent theory on the rupture (cavitation) for liquids, common peculiarities of the failure conditions for both classes of materials are presented. Finally, by introducing a stress field parameter, the stress triaxiality t, which can be used to identify the limit condition of collapse for both classes of the considered materials, some unified considerations on failure of solids and liquids are made. It is shown that the collapse functions for both classes of continuum materials present a similar form.

KEYWORDS. Failure; Fracture; Cavitation; Plasticity; Stress Triaxiality; Solid; Liquids.

INTRODUCTION

Failure in continuum media (solids or liquids) can be regarded as a physical phenomenon which occurs when a limit condition depending on the stress field is fulfilled. In isotropic media, failure conditions can conveniently be written in terms of the stress invariants and deviatoric stress invariants. By analysing the failure of ductile or brittle solids, the classical theory of plasticity and fracture mechanics concepts can be used, respectively, whereas the occurrence of the so-called cavitation must be examined for the failure of liquids.

In the present paper, a unified discussion about such failures mechanisms shows that the collapse formulation is similar for both classes of continuum materials (solids and liquids). By introducing a stress field parameter (the stress triaxiality), the limit conditions of collapse for both the above classes can simply be represented through the same equation by properly setting the involved material parameters, the values of which give us useful information about the kind of expected failure.

Local collapse of continuum media, both solids (mainly brittle solids) and liquids, frequently appears as a loss of continuity of the material. The above phenomenon occurs through the formation of cracks (fracture collapse) or high strain deformation zones (due to plastic flow) in solids, or through the appearance of voids or bubbles (due to the cavitation phenomenon) in liquids.

Collapse of solids has been widely analysed in the literature. The classical plasticity theory [1, 2] has been applied to recent studies on crystal plasticity [3, 4]. On the other hand, fracture mechanics concepts, developed from the pioneering researches carried out by Griffith [5, 6] and Westergaard [7] up to subsequent works by Liebowitz [8] and Sih [9], have been applied to recent studies on fracture at microscopic level and nanoscale level [10, 11]. Liquid collapse, also called liquid cavitation, has been studied by several Authors [12-18]. By adopting a stress-based approach to describe the failure phenomena in solids and liquids [4, 12], similar governing equations can be found.

FAILURE IN SOLIDS

Failure in solids can be regarded as a phenomenon which involves an irreversible re-arrangement of its micro-structure accompanied by a dissipation of energy. Such a modification of the material structure can appear either as a plastic flow due to the sliding - at the microscopic scale - of the lattice structure of the material, characterised by high plastic strain level (typical of the so-called ductile materials) or as a continuity loss due to a high strain localization (as usually occurs in brittle materials), leading to material separation (crack formation).

Fracture in solids usually takes place due to the growth and coalescence of micro-cracks or voids, produced by the high stress state level in the material, up to the formation of a macro-defect which can mathematically be described as a strain localisation in a narrow band which identifies the fracture position.

Typical approaches to examine the fracture phenomenon are based on either the Stress-Intensity Factor (SIF) value (descending from the classical Westergaard solution in linear elastic fracture mechanics, LEFM [7]) or the energy criterion proposed by Griffith [5, 6].

The criteria to describe catastrophic fracture occurrence under pure Mode I can be written by the following inequalities related to the critical SIF (K_{IC}) and the energy approach, respectively [5, 6]:

$$K_I = Y_I \cdot \sigma_0 \sqrt{\pi \cdot a} \geq K_{IC}, \quad \mathsf{G} = \pi \cdot a \cdot {\sigma_0}^2 / E \geq \mathsf{G}_{IC} \tag{1}$$

where Y_I is the so-called geometric correction factor (or dimensionless SIF), σ_0 is the applied remote stress, a is a characteristic size of the existing crack, G is the fracture energy, K_{IC} and G_{IC} are the fracture toughness (critical SIF) and the critical fracture energy at given environmental conditions, respectively.

The two critical parameters can be joined together by the classical relationships:

$$\mathsf{G}_{IC} = K_{IC}^2 / E \ \text{(plane stress)} \quad \mathsf{G}_{IC} = \left(1-\nu^2\right) \cdot K_{IC}^2 / E \ \text{(plane strain)} \tag{2}$$

The above fracture mechanics approaches assume that the fracture process starts from a pre-existing crack, with characteristic size a, which instantaneously grows when the critical condition (Eq.1_a or Eq.1_b) is fulfilled. In the case of absence of any defect, fracture mechanics erroneously states that no failure occurs, whatever the stress level in the body.

In macroscopically non-damaged solids, the failure process can also be regarded as a breaking or a decohesion phenomenon, which occurs when a sufficiently high stress level is greater than the tensile strength f_t of the material. The failure stress, in a general case, can be identified by one of the following conditions:

$$\sigma_0 \geq \min\left(f_t, \sqrt{\frac{E \cdot \mathsf{G}_{IC}}{\pi \cdot a}} \right) \quad \text{or} \quad \sigma_0 \geq \min\left(f_t, \frac{K_{IC}}{Y \cdot \sqrt{\pi \cdot a}} \right) \tag{3}$$

where the remote stress σ_0 is applied to the solid. From such expressions, failure-type can be identified through the characteristic flaw size a_0:

$$a_0 \begin{cases} \geq \dfrac{E \cdot \mathsf{G}_{IC}}{\pi \cdot f_t^2} & \text{fracture - like collapse} \\ < \dfrac{E \cdot \mathsf{G}_{IC}}{\pi \cdot f_t^2} & \text{plastic - like collapse} \end{cases} \tag{4}$$

In the context of plasticity, a general and common failure criterion, which has a wide applicability for several materials, is the so-called Drucker criterion [2], usually expressed by means of the stress invariants as follows:

$$I_1 + \xi \cdot \sqrt{|J_2|} - k \leq 0 \tag{5}$$

where ξ, k are material parameters that can be determined from the uniaxial tensile and compression strengths, f_t and f_c, of the material. For compressive hydrostatic-insensitive materials, the parameter ξ is approximately equal to

$-3\cdot\sqrt{3}$ and k is equal to $-3\sigma_p$ (with σ_p the hydrostatic failure compressive stress), and the criterion (5) becomes $I_1 - 3\sqrt{3}\cdot\sqrt{|J_2|} \le -3\sigma_p$ or equivalently:

$$-\kappa + \sqrt{3}\cdot\sqrt{|J_2|} \le \sigma_p \text{ with } \kappa = tr\,\boldsymbol{\sigma}/3 = \sigma_{ii}/3 \;\; (\kappa = tr\,\boldsymbol{\sigma}/2) \text{ the 3D (2D) hydrostatic pressure} \tag{6}$$

Among the various parameters to identify the '*stress state quality*' in a point of a continuum, the stress triaxiality factor t - that is, the ratio between the mean hydrostatic stress and a quantity that measures the root of the distortional energy - can be used [19-23]:

$$t = \frac{1}{3} tr\,\boldsymbol{\sigma} / \sqrt{|J_2|} = \kappa / \sqrt{|J_2|} = I_1 / 3\sqrt{|J_2|} \tag{7}$$

It can be deduced that $t \to \infty$ for purely hydrostatic stress states, $t = 0$ for purely shear stress states, $t = 1/\sqrt{3}$ for a uniaxial 3D stress state and $t = 1$ for a uniaxial 2D stress state. By using such a definition, the condition (3) for 2D problems can be written as follows:

$$\sqrt{|J_2|}\cdot(1+t) \ge \min\left(f_t, \sqrt{\frac{E\cdot G_{IC}}{\pi\cdot a}} \right) \quad \text{or} \quad \sqrt{|J_2|}\cdot(1+t) \ge \min\left(f_t, \frac{K_{IC}}{Y\cdot\sqrt{\pi\cdot a}} \right) \tag{8}$$

where the remote maximum principal stress has been written in this way: $\sigma_1 = \sqrt{|J_2|}\cdot(1+t)$. As a matter of fact:

$$\sigma_1 = S_1 + \kappa = \sqrt{|J_2|} + \kappa = \sqrt{|J_2|} + t\sqrt{|J_2|} = \sqrt{|J_2|}\cdot(1+t) \tag{9}$$

where S_1 is the principal stress of the deviatoric stress tensor.
On the other hand, by considering the Drucker plasticity collapse criterion, we get:

$$\sqrt{|J_2|}\cdot\left(\frac{\xi}{2} + t\right) \ge \frac{k}{2} \tag{10}$$

and, in the particular case expressed by Eq. (6), the criterion becomes:

$$\sqrt{|J_2|}\cdot(-\sqrt{3} + t) \ge \sigma_p \tag{11}$$

The above failure expressions (8, 10) can be summarised in the following general equation:

$$\sqrt{|J_2|}\cdot(\pm t + \beta) \ge B \tag{12}$$

where β, B are constants related to material behaviour, as is discussed below.
It can be observed [24-28] that the stress triaxiality represents a simple parameter to identify the type of collapse: stress triaxiality lower than zero ($t = 0$, pure shear) produces a shear-type plastic flow collapse, which can be identified as a special case of the fracture process, while positive triaxiality produces a void formation-type fracture process (fracture along a direction normal to the principal stress) [27, 28]. Experimental observations [21] have shown that fracture never occurs for $t \le -1/3$.

FAILURE IN LIQUIDS

It is a common observation that liquid failure takes place by the appearance of voids or bubbles with loss of continuity and, as is well-known, with the corrosion of solid in contact which such broken liquid zones.
The knowledge of the state of stress in a liquid is fundamental to evaluate the possibility of its breaking. The stress state $\boldsymbol{\sigma} = \sigma_{ij}$ can generally be written as follows:

$$\boldsymbol{\sigma} = -p\cdot\mathbf{1} + \boldsymbol{\tau}[\mathbf{u}] \tag{13}$$

where p is a scalar that represents the hydrostatic pressure (typically a positive value of p indicates hydrostatic compression in liquids), $\mathbf{1}$ is the identity tensor, and $\boldsymbol{\tau}$ is the shearing stress tensor which can be determined by an appropriate constitutive law depending on the liquid under consideration. Usually, the shearing stress tensor $\boldsymbol{\tau}$ can be written as an appropriate function of the velocity, as in the simple linear constitutive equation for a Newtonian liquid: $\boldsymbol{\tau}[\mathbf{u}] = 2\mu \cdot \mathbf{D}[\mathbf{u}]$, where μ is the dynamic viscosity, and $\mathbf{D}[\mathbf{u}] = D_{ij} = 1/2(\nabla\mathbf{u} + \nabla^T\mathbf{u}) = 1/2(u_{i,j} + u_{j,i})$ is the symmetric part of the gradient of the velocity vector u_i, $i = 1,2,3$. As is well-known, for an incompressible liquid: $div\,\mathbf{u} = tr\,\mathbf{D}[\mathbf{u}] = u_{i,i} = 0$ (mass conservation equation). From the previous equation, it can be stated that $tr\,\boldsymbol{\tau} = tr\,(2\mu \cdot \mathbf{D}[\mathbf{u}]) = 0$, $\tau_{ij} = \mu \cdot (u_{i,j} + u_{j,i})$, and the mean hydrostatic pressure becomes: $p = -\sigma_{ii}/3$. The above equation also holds for a liquid in a static condition ($\mathbf{D}[\mathbf{u}] = 0$) for which the hydrostatic pressure p represents the average of the principal stresses.

In a flowing liquid, the hydrostatic pressure is $p = -1/3 \cdot tr\,(\boldsymbol{\sigma} - \boldsymbol{\tau})$ since $tr\,\boldsymbol{\tau} \neq 0$ for compressible or non-Newtonian liquids [16, 18]. In this case, Eq. (13) can be rewritten as follows:

$$\boldsymbol{\sigma} = -p \cdot \mathbf{1} + \boldsymbol{\tau} = \kappa \cdot \mathbf{1} + \mathbf{S} \tag{14}$$

where $p = -1/3 \cdot tr\,(\boldsymbol{\sigma} - \boldsymbol{\tau})$, $\kappa = 1/3\,tr\,\boldsymbol{\sigma} = -p - 1/3\,tr\,\boldsymbol{\tau}$, and $\mathbf{S}$ is the deviatoric stress tensor for which $tr\,\mathbf{S} = s_{ii} = 0$ holds (in liquids, the scalars p and κ are assumed positive for hydrostatic compression stress states).

The failure phenomenon in a liquid is known as cavitation, and takes place by the formation of bubbles. Such a phenomenon is classically identified by the following condition [18]:

$$p \begin{cases} < p_c(T) & \text{cavitation} \\ > p_c(T) & \text{no cavitation} \end{cases} \tag{15}$$

where $p_c(T)$ is the cavitation threshold pressure (negative, i.e. tension in liquids) of the examined liquid at a given temperature T. For liquids in motion, the above condition can be written by taking into account the principal stresses instead of the average pressure p which is meaningless in such a case [18]. The principal stresses in the coordinate system coincident with the principal stress directions can be determined (the stress tensor has a diagonal form in such a coordinate system):

$$\mathrm{diag}\,\boldsymbol{\sigma} = \mathrm{diag}\,(\kappa \cdot \mathbf{1} + \mathbf{S}) = \kappa \cdot \mathbf{1} + \mathrm{diag}\,\mathbf{S} \quad \text{or} \quad \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \kappa \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathrm{diag}\,\mathbf{S} \tag{16}$$

In 2D problems, we have the following expression:

$$\mathrm{diag}\,\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \kappa \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix} = \kappa \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} S_{11} & 0 \\ 0 & -S_{11} \end{bmatrix} \quad \text{since} \quad S_{11} + S_{22} = 0 \tag{17}$$

where $\kappa = 1/2\,tr\,\boldsymbol{\sigma}$, the principal stresses of the deviatoric tensor $\mathbf{S}$ are equal to $S_{11} = +(s_{11}^2 + s_{12}^2)^{1/2}$, $S_{22} = -(s_{11}^2 + s_{12}^2)^{1/2}$ (since $s_{11} = -s_{22}$), and the term $s_{ij} = \sigma_{ij} + \delta_{ij} \cdot \kappa$ is the generic component of $\mathbf{S}$ (δ_{ij} being the Kroneker delta function). Since the principal stresses of the deviatoric tensor $\mathbf{S}$ are: $S_{11} = \sigma_1 - (\sigma_1 + \sigma_2)/2$, $S_{22} = \sigma_2 - (\sigma_1 + \sigma_2)/2$ and the maximum difference between the maximum (σ_1) and the minimum (σ_2) principal stress is equal to $\sigma_1 - \sigma_2 = S_{11} - S_{22} = 2 \cdot S_{11}$, the rupture condition (Eq.(15)), formulated in terms of the maximum principal stress, becomes:

$$\begin{aligned} &\sigma_1 \geq |p_c| \quad \text{or} \\ &\sigma_1 = S_{11} + \kappa = (s_{11}^2 + s_{12}^2)^{1/2} - \kappa = \mu \cdot \left[u_{1,1}^2 + (u_{1,2} + u_{2,1})^2\right]^{1/2} + \kappa \geq |p_c| \end{aligned} \tag{18}$$

When the above condition is fulfilled, a slit vacuum cavity initially opens in a direction perpendicular to the maximum tension axis. Subsequently, the flow vorticity can rotate the major axis of the ellipse (see Fig. 1), and vapour fills the created cavity [18]. If also $\sigma_2 = S_{22} - \kappa \geq |p_c|$, then the cavity opens also along the other principal direction.

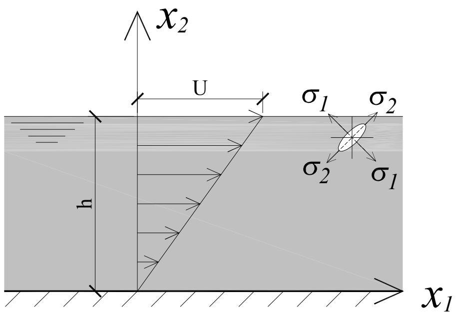

Figure 1: Scheme of the cavitation phenomenon in a liquid under shear flow.

The above criterion must be corrected in presence of impurities (such as pre existing bubbles, impurities, etc.) which facilitate the cavitation phenomenon. If a 'defect' with a characteristic size a is supposed to exist in a point of the liquid, then Eq. (15) can be written as follows [16]:

$$\sigma_1 \geq |p_c(a,T)| = |K_c(T)| / \sqrt{a} < |p_c(T)| \tag{19}$$

where $K_c(T)$ is a material parameter for a given temperature T.
Similar to the case of collapse in solids, the cavitation phenomenon in general takes place when the following condition is fulfilled [16]:

$$\sigma_1 \geq \min\left(|p_c(T)|, \frac{|K_c(T)|}{\sqrt{a}} \right) = \frac{|K_c(T)|}{\sqrt{a}} \tag{20}$$

Defining the stress triaxiality factor analogously to the case of solids (see Section 'Failure in solids'): $t = \kappa / \sqrt{|J_2|}$, with $J_2 = s_{11} \cdot s_{22} - s_{12}^2$, the maximum principal stress can be rewritten as follows: $\sigma_1 = (s_{11}^2 + s_{12}^2)^{1/2} + \kappa =$ $= (-s_{11} \cdot s_{22} + s_{12}^2)^{1/2} + \kappa = \sqrt{|J_2|} - \kappa$, and the cavitation phenomenon takes place when

$$\sigma_1 = \sqrt{|J_2|} \cdot (1+t) \geq |p_c| \quad \text{or} \quad \sigma_1 = \sqrt{|J_2|} \cdot (1-t) \geq |p_c| \quad \text{by defining as negative hydrostatic tension} \tag{21}$$

As can be observed also under pure shear ($t = 0$), liquid rupture can occur when $\sqrt{|J_2|} \geq |p_c|$. If a high level of hydrostatic compressive pressure exists ($t >> 0$, since $\kappa > 0$ for hydrostatic compression in liquids), $\sigma_1 = \sqrt{|J_2|} \cdot (1-t) < 0$ and cavitation never occurs. Some examples of fracture collapse in solids and fluids are given in Fig. 2.

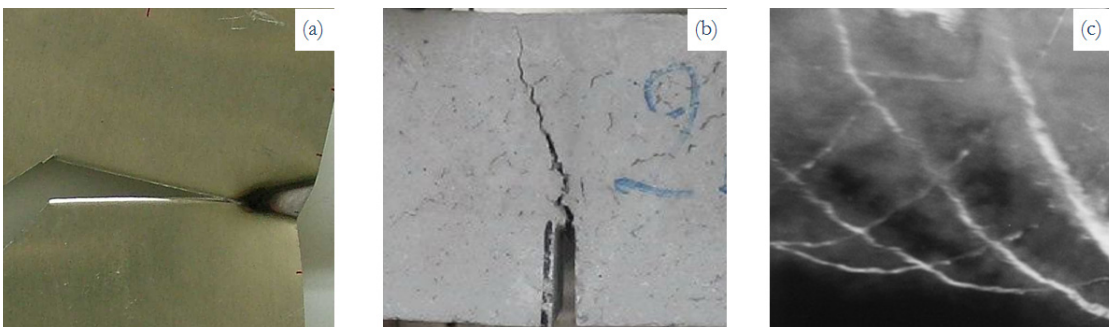

Figure 2: Fracture collapse in (a) ductile or (b) brittle solid material, and (c) rupture (cavitation) in a liquid

DISCUSSION

From the previous Sections, we can deduce that the failure condition for unflawed or flawed solids (see Eqs (3, 10)) and liquids (see Eqs (15, 20)) presents a mathematical structure which can be written as follows:

$$\sigma_1 = \sqrt{|J_2|} \cdot (1+t) \geq \min(C_1, C_2) \quad \text{where } C_1, C_2 = \begin{cases} f_t, \sqrt{\dfrac{E \cdot \mathsf{G}_{IC}}{\pi \cdot a}} = \dfrac{K_{IC}}{Y \cdot \sqrt{\pi \cdot a}} & \text{for solids} \\ |p_c(T)|, \dfrac{|K_c(T)|}{\sqrt{a}} & \text{for liquids} \end{cases} \tag{22}$$

where the same definition of κ is used for both solids and liquids, i.e. $\kappa = 1/2\, tr\, \boldsymbol{\sigma}$. In the above general relationship, a difference in the meaning of the stress σ_1 must be taken into account: the failure in liquids takes place in the point where the principal stress attains the maximum value (the imperfections are assumed to be equally distributed in the liquid domain), whereas the stress σ_1 in solids must be interpreted as the reference stress (applied to the structure) that produces the critical value of SIF.

Further, it can also be remarked that the failure conditions in unflawed solids (Eq. (10)) and liquids (Eq. (15)) have a similar form which can be summarised in this way:

$$\sqrt{|J_2|} \cdot (\beta + \alpha \cdot t) \geq B \tag{23}$$

where the material constants α, β, B depend on the mechanical material behaviour and the presence of defects. The values assumed by the three parameters involved in Eq. (23) are reported in Tab. 1.

Material	***Mechanical behaviour and defects***	**Failure parameters**		
		α	β	B
Solids	Unflawed brittle	+1	+1	f_t
	Unflawed general	+1	$\xi/2$	$k/2$
	Flawed brittle	+1	+1	$\sqrt{(E \cdot \mathsf{G}_{IC})/\pi \cdot a}$
Liquids	Unflawed	-1*	+1	$\lvert p_c(T)\rvert$
	Flawed	-1*	+1	$\lvert p_c(a,T)\rvert = \lvert K_c(T)\rvert / \sqrt{a}$

Table 1: Parameters involved in the general failure function, $\sqrt{|J_2|} \cdot (\alpha \cdot t + \beta) \geq B$, for unflawed and flawed solids and liquids in 2D problems (* Eq. (21_b) has been used to define stress triaxiality sign).

The fail safe domain expressed by Eq. (23) can graphically be represented in the $t - J_2$ co-ordinate system.

Such a domain is shown for brittle and Druker-like solids (right-hand side in Fig. 3) and liquids (left-hand side in Fig. 3).

It can be observed that curves for liquid media are characterised by positive values of the stress triaxiality for the hypotheses on the hydrostatic sign (positive compressive hydrostatic stress, top horizontal axis), while the opposite holds for solids (positive tensile hydrostatic stress, bottom horizontal axis).

Several values of the material parameters have been examined for solids, while water at environmental temperature has been used for the example related to liquids ($p_c = 2.3 KPa$ at 20ºC is the maximum tensile stress at which cavitation occurs in water).

As can be observed in Fig.3, the curve trend is similar for all cases studied, and indicates that the allowable distortional energy $\sqrt{J_2}$ leading to failure, decreases by increasing the stress triaxiality t. In particular, no cavitation occurs for $t > 1$ ($t < -1$ if the stress triaxiality sign hypotheses for solids is adopted), while no rupture takes place in solids for $t < 0$.

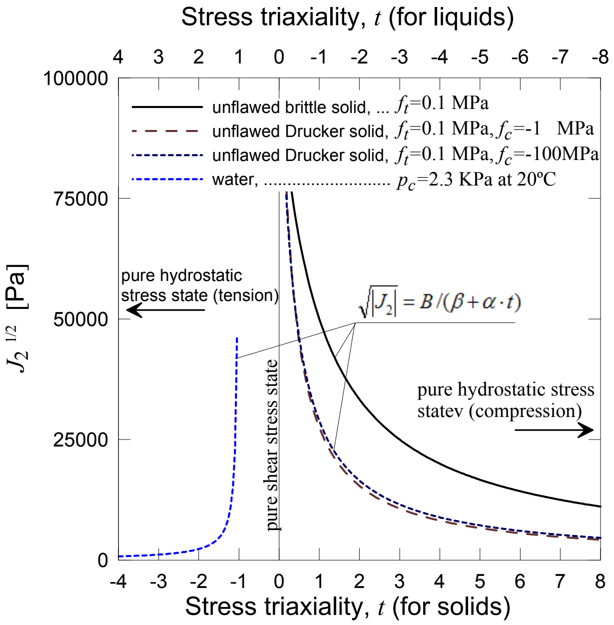

Figure 3: $J_2 - t$ relationship at collapse for unflawed solids and liquid (water). Regions below the lines $\sqrt{|J_2|}\cdot(\alpha\cdot t+\beta)=B$ represent fail-safe stress states.

CONCLUSIONS

In the present paper, failure in continuum media (solids or liquids) has been regarded as a physical phenomenon which is determined by the fulfilment of a limit condition.

By taking into account the classical plasticity or the fracture mechanics theory to describe the failure of solids and the theory of the cavitation phenomenon to describe the failure of liquids, a unified formulation has been presented, showing that collapse functions present a similar form for both classes of continuum materials (solids or liquids).

By examining 2D cases, a stress field parameter (the stress triaxiality) allows us to express the limit conditions of collapse for both classes of materials through a single equation, i.e. the structure of the governing failure equation for both solids and liquids is the same. By appropriately setting the involved material parameters, the failure function can explicitly be written. The values of such parameters can give us useful information about the kind of expected failure.

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