Sharp Notch Roots;
The length scale implicit in the solution

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ABSTRACT. Reentrant notches whose internal angle exceeds about 257° induce two singular eigensolutions, from the classical Williams procedure, where the symmetric term is always more strongly singular than the antisymmetric term. This implies the presence of a length scale within the singular solution, and it means that notch root process zones are not self-similar but vary in character according to their size; small ones will be mode-*I* like, larger ones mode-*II* like, larger ones still not existing under small scale yielding conditions. Here we explore explicitly within the framework of the singular solution (i.e. a semi-infinite notch) the conditions under which mode *I* type behaviour exists, or mode *II* behaviour, or the solution is mixed in character. These general results are then applied to example finite problems, and used to show the range of loads under which pure mode *I*, small scale yielding singular behaviour is to be expected. This is of practical relevance because it means that, even if both eigenmodes are excited in a particular example problem, the process zone may practically be considered to be mode *I* in nature. It is shown that, in each of the example problems examined so far, there is a wide range of conditions where this is so, and this property may be used to simplify the way we treat the effects of sharp corners, whether at notches or at the edges of complete contacts, as characterizers of the local process zone.

KEYWORDS: Notches; William' solution; length scale; process zone.

INTRODUCTION

The sharp elastic notch has been studied extensively and widely for fifty years, and by many people; it might be thought that there was little left to say. But the function of this paper is very simple: it is to bring out the intrinsic length dimensions present in all semi-infinite 'wedge' solutions of included angle strictly less than 360 degrees (thus excluding the crack), and to use this in a practical way to show how, under mixed mode loading conditions, it is almost invariably the symmetrical eigensolution which dominates the solution, and therefore controls crack nucleation conditions. We will do this by first re-casting the classical Williams solution [1] in a novel way, and then by using it as a characteriser of two example finite notch root problems. This will permit a clear definition of when the process zone is controlled by the mode I eigensolution alone, is mixed mode, is controlled by the mode II eigensolution predominantly, or is, indeed, not truly controlled by 'small scale yielding' (SSY) singular term considerations.

Williams' Solution

The starting point of our analysis is the celebrated analysis carried out by Williams to find the dominant terms in an eigenfunction expansion of the state of stress around a notch root [1, 2], and this will not be spelled out here. For notches whose angle, 2α, in the solid exceeds 257.4 degrees there are two singular terms, and we may write the local stresses down as:

$$\sigma_{ij}(r,\theta) = K_I r^{\lambda_I - 1} f_{ij}^I(\theta) + K_{II} r^{\lambda_{II} - 1} f_{ij}^{II}(\theta) + \text{ bounded terms}$$
$$\sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right) \tag{1}$$

where (r,θ) are defined in Fig. 1, and $0 < \lambda_I < \lambda_{II} < 1$. Because of the different powers attributable to the symmetric (λ_I) and antisymmetric (λ_{II}) terms, there is an inherent length scale in this semi-infinite solution, which we will call d_0, and define this as:

$$d_0 = \left| \frac{K_{II}}{K_I} \right|^{\frac{1}{\lambda_I - \lambda_{II}}} \tag{2}$$

We may use this as a non-dimensionalising quantity for the radial coordinate by dividing through by this quantity, and restore symmetry to the relative contributions provided by each mode of loading by introducing a new quantity describing the magnitude of the loading, G_0, the 'notch extension force', given by:

$$G_0 = K_I^{\frac{\lambda_{II} - 1}{\lambda_{II} - \lambda_I}} K_{II}^{\frac{\lambda_I - 1}{\lambda_I - \lambda_{II}}} \tag{3}$$

So that, finally:

$$\frac{\sigma_{ij}(r,\theta)}{G_0} = \left(\frac{r}{d_0}\right)^{\lambda_I - 1} f_{ij}^I(\theta) + \left(\frac{r}{d_0}\right)^{\lambda_{II} - 1} f_{ij}^{II}(\theta) \tag{4}$$
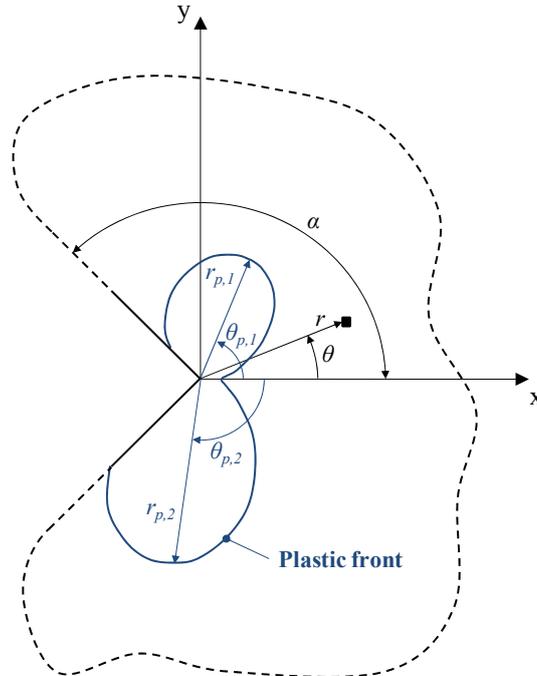

Figure 1: Schematic of a homogeneous wedge of internal angle 2α: the characteristics radii and angles corresponding to the plastic front at the semi-infinite sharp notch root are highlighted.

The quantity d_0 is seen to be a measure of mode mixity with dimensions of length. From now on in this paper we shall concentrate on the case of a 270 degree wedge so that:

$$\lambda_I = 0.5445, \quad \lambda_{II} = 0.9085 \tag{5}$$

and show, in Fig. 2, the shape and size of the process zone, $r_p(\theta)/d_0$, as a function of the applied load, G_0/k, where k is the yield stress in pure shear. The important feature of the solution which becomes apparent is that, if the observation point is much closer to the notch corner than d_0 ($r \ll d_0$) the stress field is essentially mode I in character, whilst if it is much more remote ($r \gg d_0$) it is clearly mode *II* in character. When the observation point is comparable with d_0 ($r \sim d_0$) it is mixed mode. Thus, in qualitative terms, for a given pair of values (K_I, K_{II}) a very strong material producing a small process zone will experience a mode *I* type field, and a very soft material, having a very much larger process zone will experience a mode *II* type field. To summarise this observation more quantitatively, a further calculation of the characteristics of the plastic zone lobes was conducted. A radial line is drawn from the origin to a point defining the extent of the lobe (r_{max}), and the angle this makes with the notch bisector found. The results are shown in Fig. 3. Thus if $\theta \sim \pi/2$ the loading is mode I in character whilst if $\theta \sim 0$, it is mode *II*. This shows that, for the process zone to be mode *I* we must have $G_0/k < 0.1$, and for it to be mode *II* we must have $G_0/k > 1.2$. In order to give further physical interpretation to these properties we must attach the semi-infinite solution to a typical finite notch problem.
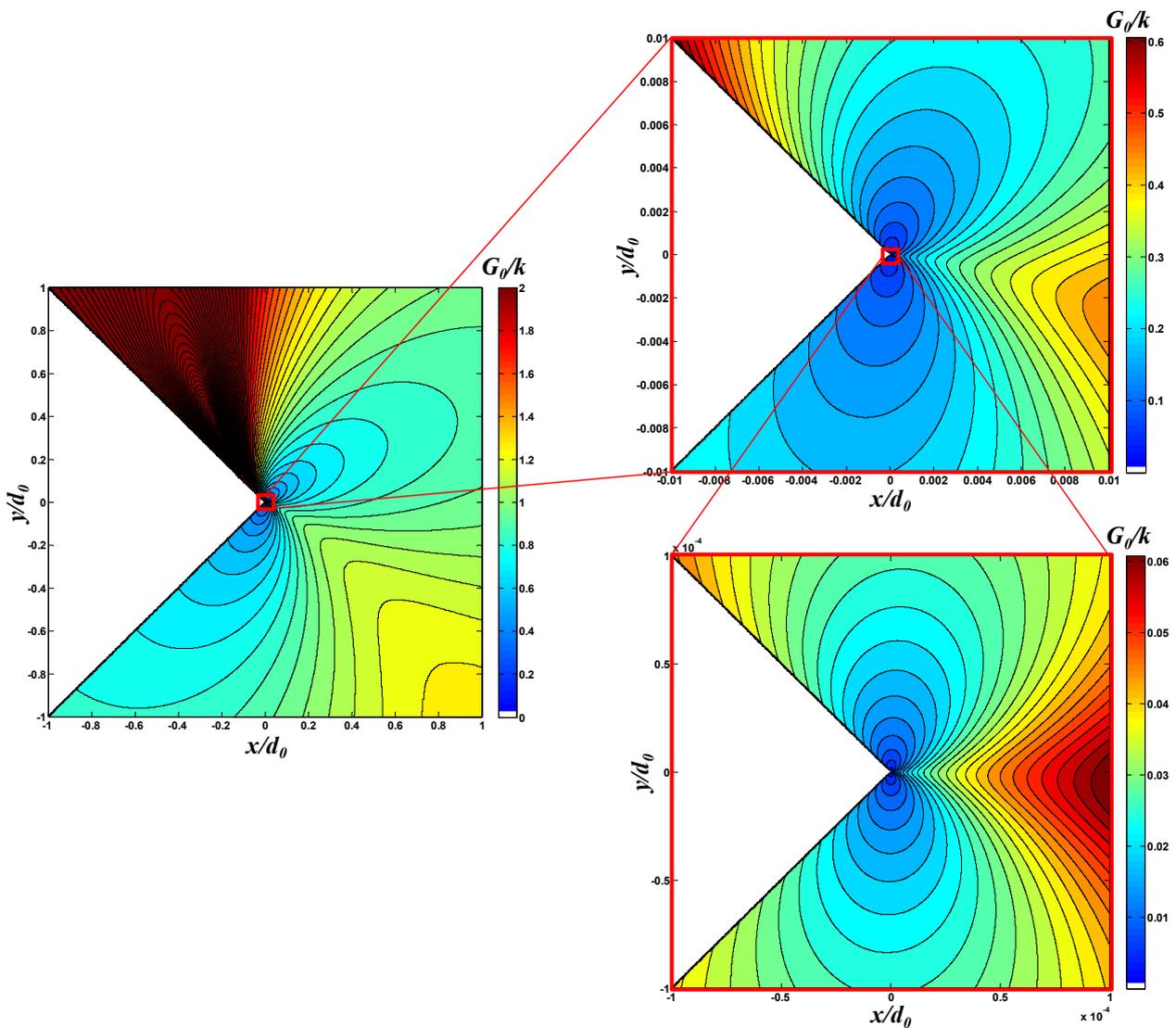

Figure 2: Contours of the estimated size of the plastic fronts $r_p(\theta)/d_0$, as a function of the dimensionless load G_0/k, at three different levels of magnification for a $2\alpha=270°$ wedge.
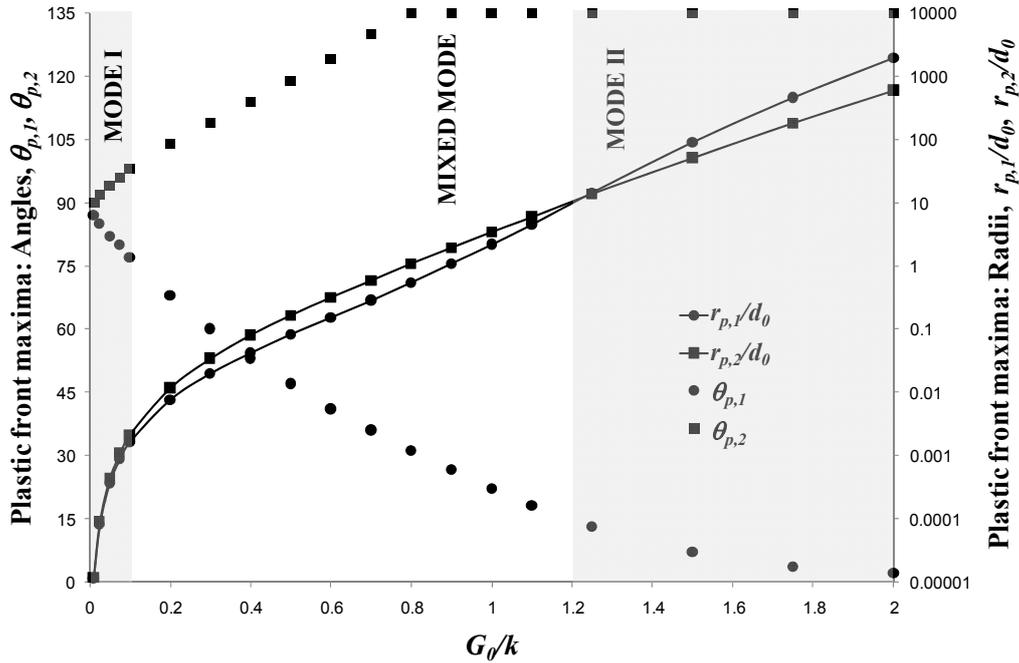
Figure 3: Plastic "lobes" characteristics: normalised plastic front maximum extents, $r_{p,1}/d_0$ and $r_{p,2}/d_0$ and corresponding angles, $\theta_{p,1}$ and $\theta_{p,1}$, as a function of the normalised notch extension force, G_0/k for a $2\alpha = 270°$ wedge.

EXAMPLE PROBLEM: SLANT EDGE NOTCH

The problem chosen is a half-plane, including an edge notch of solid angle 270 degrees, and where the notch centreline makes an angle of 30 degrees to the edge normal, Fig. 4(a). If σ_0 is the remote tension, and 'a' the notch depth, the stress intensity factors may be written as:

$$K_i = \Lambda_i \sigma_o a^{1-\lambda_i} \qquad i = I, II \tag{6}$$

and we find, from a routine Abaqus commercial FEM model, that

$$\Lambda_I = 0.924, \ \Lambda_{II} = 0.327 \tag{7}$$

and, hence:

$$d_0/a = 17.3, \ G_0/\sigma_0 = 0.252 \tag{8}$$

This immediately shows that, for this particular geometry, because the mode *I* / mode *II* transition length is very much greater than the notch length, it would be impossible to have a mode *II* type process zone and retain the assumption of small scale yielding. It follows that, for this particular geometry, crack nucleation for a slant notch, and therefore apparently experiencing mixed mode loading, will, if small scale yielding is maintained, be mode *I* in character.

SECOND EXAMPLE: SQUARE HOLE IN INFINITE PLATE

It is clear, in the above example, that most practical types of remote loading will, at realistic loads, give rise to mode I notch root stress fields for 90 degree internal corners. The question arises of whether it is possible to attain mode II notch root conditions with more extreme types of far field load. To this end, consider the problem of an infinite

plate with a square cut-out, of diagonal length a. It is oriented as shown in Fig. 4(b), so that a remote tension, σ_0, induces only mode I loading of the hole corners, and a remote shear stress, τ_0 induces only shear. A simple finite element model shows the calibrations for the stress intensity factors to be:

$$\begin{Bmatrix} K_I a^{\lambda_I - 1} \\ K_{II} a^{\lambda_{II} - 1} \end{Bmatrix} = \begin{bmatrix} 0.477 & 0 \\ 0 & 1.451 \end{bmatrix} \begin{Bmatrix} \sigma_0 \\ \tau_0 \end{Bmatrix} \tag{9}$$

We now use the hole size as the reference length scale and specialise the radius to the plastic boundary, r_p, to give:

$$\frac{r_p}{a} = \frac{r_p}{d_0}\frac{d_0}{a} = \frac{r_p}{d_0}\frac{1}{a}\left|\frac{K_{II}}{K_I}\right|^{\frac{1}{\lambda_I - \lambda_{II}}} = \frac{r_p}{d_0}\left|\frac{1.451\tau_o}{0.477\sigma_o}\right|^{-2.747} = \frac{r_p}{d_0}\left|\frac{0.477\sigma_o}{1.451\tau_o}\right|^{2.747} = 0.0471\frac{r_p}{d_0}\left[\frac{\sigma_o}{\tau_o}\right]^{2.747} \tag{10}$$

and we can now write down the generalised notch extension force as:

$$\frac{G_0}{\sigma_0} = \frac{K_I^{\frac{\lambda_{II}-1}{\lambda_{II}-\lambda_I}} K_{II}^{\frac{\lambda_I-1}{\lambda_I-\lambda_{II}}}}{\sigma_0} = \left[0.477\right]^{\frac{\lambda_{II}-1}{\lambda_{II}-\lambda_I}}\left[1.451\frac{\tau_0}{\sigma_0}\right]^{\frac{\lambda_I-1}{\lambda_I-\lambda_{II}}} = 1.919\left(\frac{\tau_0}{\sigma_0}\right)^{1.25125} \tag{11}$$

or

$$\frac{G_0}{k} = 1.919\left(\frac{\tau_0}{k}\right)^{1.25125}\left(\frac{k}{\sigma_0}\right)^{0.2515} \tag{12}$$

These results enable us to establish the boundaries for the different quantities dominating control of the process zone, for this particular problem; here, we know that, because the excitation of the two modes of loading is uncoupled, any mix of remote loading may be applied, and any magnitude, so it should be possible to provoke any solution between small scale yielding in notional mode I or mode II, small scale yielding but mixed in character, and when the process zone generally falls outside the domain dominated by the William's solution. The outcome is displayed in Fig. 5, where the axes are the dimensionless remote loads σ_0/k, τ_0/k.

The first step is to establish the mode I dominant and mode II dominant boundaries by setting G_0/k to 0.1 and 1.2. We now establish the limits of validity of the William's two-term solution (small scale yielding) by considering pairs of applied load (σ_0/k, τ_0/k), and finding (G_0/k). Then, from Fig. 3, we determine the maximum size of the plastic zone (r_p/d_0), and hence (r_p/a). We then need to estimate what we can accept as a possible maximum value of this quantity, and we have arbitrarily chosen 1% and 5% as sensible choices.

For comparison, we have also calculated the collapse load for the problem, which corresponds, here, to the entire plane becoming plastic. Under plane strain conditions the limit state is given by:

$$\left(\frac{\tau_o}{k}\right)^2 + \frac{(1 + \nu + \nu^2)}{3}\left(\frac{\sigma_o}{k}\right)^2 = 1 \tag{13}$$

Fig. 5 is, in effect, a map describing how the magnitude and mix of remote loading can put the hole corners into characteristic stress, and hence strain states, and which would, under cyclic loading cause crack nucleation. In very rough terms indeed the requirement that the cutout corners be mode I in character is that ($\sigma_0/k < 0.3$, $\tau_0/k < 0.05$): in turn, this shows that up to about 16% remote shear can be added in to the solution at light loads but the hole corners will still 'see' virtually a pure mode I loading at the distances important for irreversible plastic strains of this type to control crack nucleation.
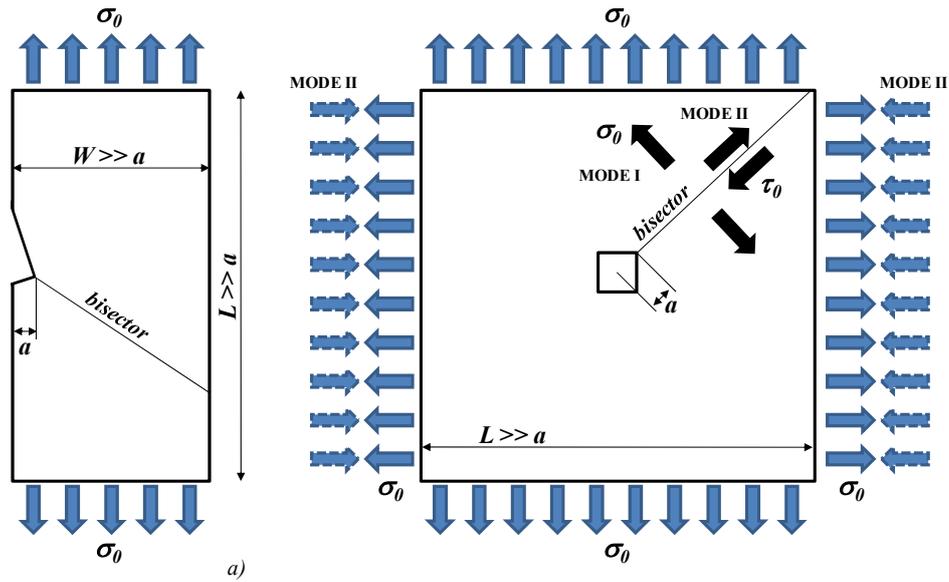
Figure 4: Schematic of the two example problems: (a) oblique edge notch; (b) a square hole in an infinite plane.
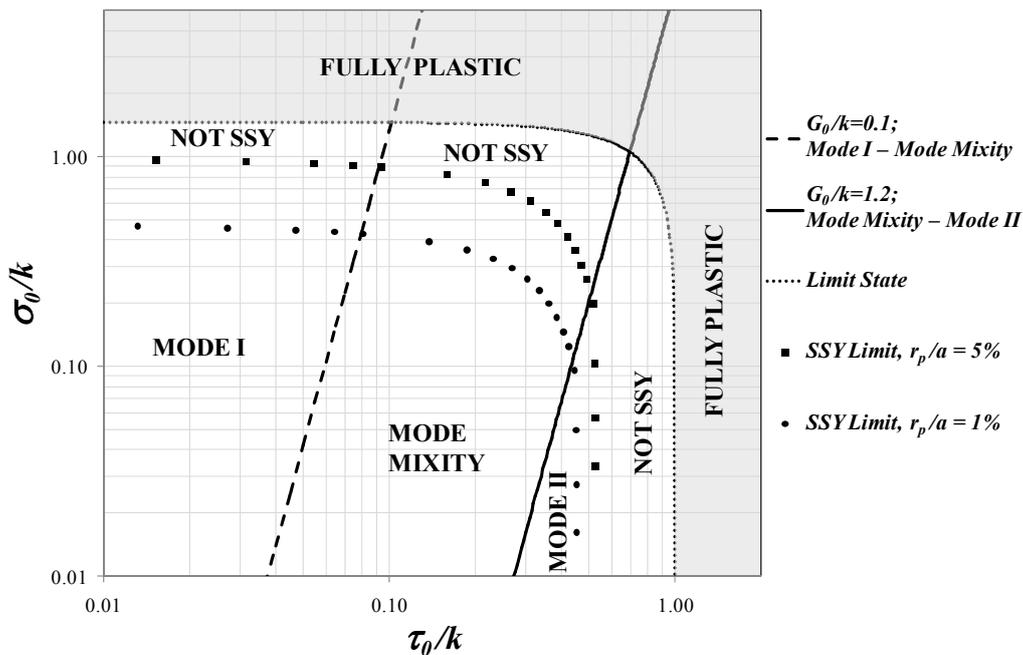

Figure 5: Map of the Mode I, Mode Mixity and Mode II regimes in the σ_0/k-τ_0/k space as implied by the semi-infinite solution for the example case in Fig. 4(b). The SSY thresholds for two values of the ratio r_p/a and the limit state transition to fully plastic behaviour are also shown.

CONCLUSION

The progress made in this paper is, first, a re-casting of the classical Williams solution so as to bring out the parameters (a) driving the magnitude of the stresses and (b) controlling the local mixity. This is useful because, when the notch-root process zone is small ($r_p \ll d_0$) it is mode I (symmetrical) in character and if ($r_p \gg d_0$) it is mode II (unsymmetrical) in character. Further, when pasted into typical problems the results enable the forms of remote loading which, in particular, lead to a strong mode I dominant solution to be established. This is of practical significance because

it enables a reduced (single variable) characterisation of the process zone to be established, and therefore reduces the amount of mechanical testing needed.

ACKNOWLEDGMENTS

REFERENCES

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