Modal decomposition of instabilities in thin-walled structural elements

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ABSTRACT. The paper presents the development of a new method for the decomposition of any deformed shape of a buckled thin-walled structural member into the contributions of a number of predefined classes of modes. In this method, these classes encompass five types, namely the local, distortional, global, transverse extension and shear modes. The modal decomposition process involves two steps. First, the transverse extension modes are separated out by enforcing the null-transverse-stress criterion. Second, the basis vectors of the local, distortional, global and shear spaces are obtained by either minimizing or maximizing the contributions of the bending energy and membrane energy in the total strain energy.

KEYWORDS. Thin-walled; Modal decomposition; Buckling modes.

INTRODUCTION

Thin-walled steel structures are enjoying an ever increasing popularity in the construction industry due to their high strength-to-weight ratio, straightforward manufacturing process and unrivalled speed of construction. Despite their appealing advantages, they are highly susceptible to instabilities due to their limited wall thickness. These instabilities, which in their 'pure' forms are often categorized as local, distortional and global modes, usually occur in combinations with significant interaction. For the purpose of both theoretical study and design, however, it is desirable to identify the contributions of the pure modes in the failure mechanism. Over the past decades two methods of modal decomposition have been proposed, which include Generalised Beam Theory (GBT) and the Constrained Finite Strip Method (cFSM). GBT was initially introduced by Schardt, R. [1,2] and can be seen as an extension of the classical theories of bending and torsion to include local and distortional deformations. Partially inspired by GBT, Ádány and Schafer [3-8] developed the cFSM, which is a modal decomposition method based on the Finite Strip Method (FSM). In this paper, a new modal decomposition method is proposed, which does not rely on the idealized assumptions of GBT. Although more generally valid, the method is here implemented in the framework of the FSM.

FSM ESSENTIALS

The finite strip method (FSM) can be seen as a specialised finite element method (FEM) dealing with prismatic thin-walled structural members. In this method, only cross-sectional discretisation is required, while sinusoidal shape functions are used longitudinally. At the minimum level, the cross-section is discretised into flat plates at the main nodes, although each flat plate can be further discretised into smaller strips using sub nodes to improve accuracy.

Main nodes associated with only one strip are called external main nodes, as opposed to internal main nodes, which connect two or more strips. These definitions are illustrated in Figure 1.

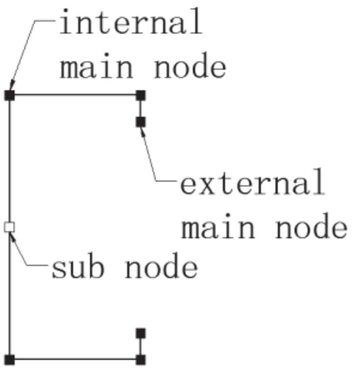
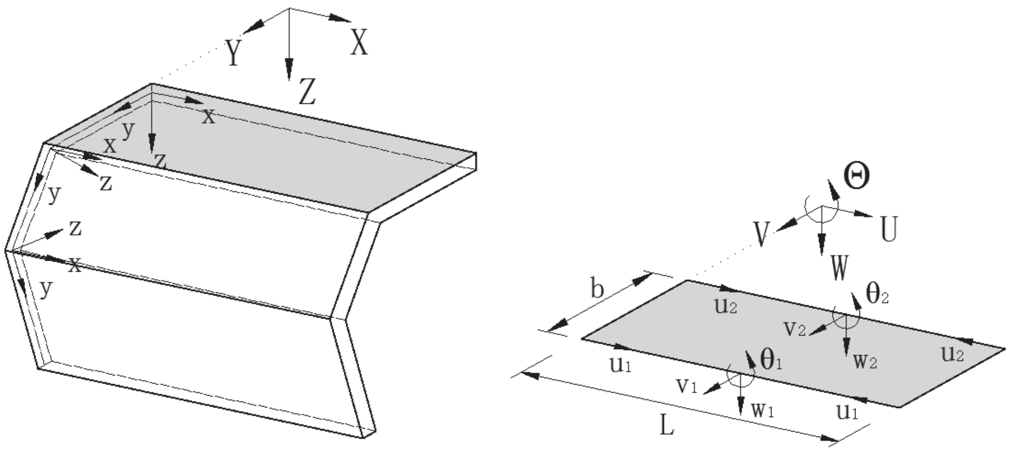


Figure 1: Definitions of node types.

In the FSM, a global coordinate system is defined, as well as a local coordinate system associated with each strip, as shown in Figure 2 (a).



(a) (b)

Figure 2: Coordinate systems and degrees of freedom.

In the local coordinate system of each strip, each nodal line has four degrees of freedom (DOF): the longitudinal and transverse in-plane displacements u and v, the out-of-plane displacement w and the out-of-plane rotation (about the local x-axis) θ. In the global coordinate system, the corresponding DOFs are the longitudinal end displacement U, the translations V and W in the global Y and Z directions and the rotation about the X axis Θ. Figure 2(b) shows an isolated strip with its degrees of freedom. L is the length of the strip (and the member), while b is the strip width.
In the FSM, the buckling modes $\boldsymbol{d}$ of the member (which may be coupled modes rather than pure modes) are obtained by solving the following eigenvalue problem:

$$(\boldsymbol{K} - \lambda \boldsymbol{G})\boldsymbol{d} = 0 \tag{1}$$

where the eigenvalues λ are proportional to the buckling stresses and $\boldsymbol{K}$ and $\boldsymbol{G}$ are the global elastic and geometric stiffness matrices, respectively.

MODAL DECOMPOSITION

The modal decomposition process involves two steps. First, the transverse extension modes are separated out by enforcing the null-transverse-stress criterion. Second, within the remaining space the basis vectors of the local, distortional, global and shear spaces are obtained by either minimizing or maximizing the contributions of the bending energy versus the membrane energy in the total strain energy.

Number of modes

In this paper, (uncoupled) buckling modes are categorized into five classes: local, distortional, global, transverse extension and shear modes. Each class contains a well-defined number of modes, as listed in Table 1 for the case of an open cross-section. The total number of nodes is thereby indicated by n, while n_m is number of internal main nodes and n_s is the number of strips. The readers is referred to [9] for a detailed explanation.

Mode type	Number of modes
Local	$2n-n_m$
Distortional	$n_m - 2$
Global	3
Transverse extension	$n_s = n - 1$
Shear	n

Table 1: Number of modes.

Step 1

The null-transverse-stress criterion $\sigma_y = 0$ is used to isolate the transverse extension modes from the other modes. However, due to the particular choice of the shape functions in the FSM, this condition can only be applied 'on average' over the strip: $\bar{\sigma}_y = 0$, where:

$$\begin{aligned} \bar{\sigma}_y &= \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\bar{\varepsilon}_x\right) = 0 \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \frac{v_2 - v_1}{b}\sin\left(\frac{\pi x}{L}\right) \\ \bar{\varepsilon}_x &= \frac{\partial u}{\partial x} = -\bar{u}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right) \\ &= -\frac{u_1+u_2}{2}\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right) \end{aligned} \tag{2}$$

Written in terms of the global displacements, Eq. (2) becomes:

$$\frac{\left(V_1^i - V_2^i\right)\cos\alpha^i + \left(W_1^i - W_2^i\right)\sin\alpha^i}{b^i} + \frac{U_1^i + U_2^i}{2}\nu\left(\frac{\pi}{L}\right) = 0 \qquad i = 1\ldots n_s \tag{3}$$

where the superscript i refers to the i^{th} strip and α is the angle measured from the local y axis of the i^{th} strip to the global Y axis. Eq. (3), when written for all strips, can be summarised in matrix form as:

$$\boldsymbol{Cd} = \boldsymbol{0} \tag{4}$$

Therefore, a matrix containing a set of column basis vectors of the combined local, distortional, global and shear space is obtained by finding the null space of $\boldsymbol{C}$:

$$\boldsymbol{H}_{ldgs} = null(\boldsymbol{C}) \tag{5}$$

We wish to impose the additional requirement that all local, distortional, global, shear and transverse extension modes are orthogonal to each other, where orthogonality (by choice) is defined with respect to the $\boldsymbol{K}$ matrix. The full set of modes then constitutes an orthogonal basis of the whole deformation space. Orthogonality of the transverse extension modes with respect to the remaining modes requires:

$$\boldsymbol{H}_{ldgs}{}^{T}\boldsymbol{K}\boldsymbol{d}_{te} = \boldsymbol{0} \tag{6}$$

Thus, the matrix containing the basis vectors of the transverse extension space is obtained as follows:

$$\boldsymbol{H}_{te} = null\left(\boldsymbol{H}_{ldgs}{}^{T}\boldsymbol{K}\right) \tag{7}$$

Once the basis vectors of a space are obtained, the actual buckling modes under a given loading can be obtained from the constrained eigenvalue problem [9]:

$$\left(\boldsymbol{H}_{te}^{T}\boldsymbol{K}\boldsymbol{H}_{te} - \lambda\boldsymbol{H}_{te}^{T}\boldsymbol{G}\boldsymbol{H}_{te}\right)\boldsymbol{a} = 0 \tag{8}$$

while the transverse extension modes are given by:

$$\boldsymbol{d}_{te} = \boldsymbol{H}_{te}\boldsymbol{a} \tag{9}$$

Step 2

The total strain energy U of a deformed thin-walled member consists of two components: the plate bending energy U_B and the membrane (in-plane) deformation energy U_M. The energies U, U_B and U_M are functions of the displacements $\boldsymbol{d}$ as follows:

$$U = \frac{1}{2}\boldsymbol{d}^{T}\boldsymbol{K}\boldsymbol{d}\ ,\ U_B = \frac{1}{2}\boldsymbol{d}^{T}\boldsymbol{K}_B\boldsymbol{d}\ ,\ U_M = \frac{1}{2}\boldsymbol{d}^{T}\boldsymbol{K}_M\boldsymbol{d} \tag{10}$$

where the matrices $\boldsymbol{K}_B$ and $\boldsymbol{K}_M$ contain the bending stiffness and the membrane stiffness components of the matrix $\boldsymbol{K}$, respectively.

The decomposition method is based on the principle, first of all, that the local buckling modes almost exclusively mobilize the plate bending deformations, with near zero contributions of the membrane deformations. As a matter of fact, in idealized cross-sections exclusively comprised of straight plates with no rounded transition zones the participation of the plate bending energy in the total energy is absolute. On the other hand, the global modes (flexural, torsional and flexural-torsional modes) have the characteristic that they mostly involve membrane deformations. However, while small, the plate bending deformations are not entirely zero in the global modes, mostly as a result of Poisson's effects. It is noted that this is usually disregarded in other decomposition methods, which consider the cross-section to undergo displacements as a rigid body in the global modes. Based on the above, the local modes can be determined by maximizing the following expression, which includes the constraint (expressed by means of a Lagrange multiplier α) that the total strain energy is equal to unity:

$$\frac{1}{2}\boldsymbol{d}^T\left(\boldsymbol{K}_B-\boldsymbol{K}_M\right)\boldsymbol{d}-\alpha\left(\frac{1}{2}\boldsymbol{d}^T\boldsymbol{K}\boldsymbol{d}-1\right) \tag{11}$$

An analogous equation can be written for the global modes. Differentiating these equations with respect to the degrees of freedom contained in $\boldsymbol{d}$ leads to the new eigenvalue problem:

$$\left[\left(\boldsymbol{K}_B-\boldsymbol{K}_M\right)-\alpha\boldsymbol{K}\right]\boldsymbol{d}=\boldsymbol{0} \tag{12}$$

where the original Langrange multiplier now takes on the function of an eigenvalue. However, the earlier obtained transverse extension modes should be excluded from the deformations $\boldsymbol{d}$:

$$\boldsymbol{d}=\boldsymbol{H}_{ldgs}\boldsymbol{a} \tag{13}$$

resulting in:

$$\left[\left(\boldsymbol{K}_B^{ldgs}-\boldsymbol{K}_M^{ldgs}\right)-\alpha\boldsymbol{K}^{ldgs}\right]\boldsymbol{a}=\boldsymbol{0} \tag{14}$$

where

$$\boldsymbol{K}_B^{ldgs}=\boldsymbol{H}_{ldgs}{}^T\boldsymbol{K}_B\boldsymbol{H}_{ldgs},\ \boldsymbol{K}_M^{ldgs}=\boldsymbol{H}_{ldgs}{}^T\boldsymbol{K}_M\boldsymbol{H}_{ldgs} \text{ and } \boldsymbol{K}^{ldgs}=\boldsymbol{H}_{ldgs}{}^T\boldsymbol{K}\boldsymbol{H}_{ldgs}.$$

The basis vectors of the local space follow from Eqs. (13-14) with α values equal to (or very close to) 1, while the global basis vectors have α values close to -1 and the shear modes (consisting entirely of in-plane membrane deformations) have α values equal to -1. The distortional basis vectors display α values much closer to 0 and are implicitly determined by the orthogonality of the solutions of Eq. (14). Ranking the eigenvectors by α value, in combination with the information in Table 1, thus allows the basis vectors of the local, distortional, global and shear spaces to be determined. Once the basis vectors of a given space are determined, the actual buckling modes under a given loading are obtained by formulating a constrained eigenvalue problem of the form of Eq. (8). The participation of the various classes of buckling modes in a given deformed shape can then be determined on the basis of their strain energy contributions in the total strain energy.

ILLUSTRATIVE EXAMPLE

In order to illustrate the proposed method, the double-cell box section shown in Figure 3 is studied. The resulting buckling modes are shown in Table 2. The critical stresses and the modal participation of the various classes in the FSM output are plotted against the buckle half-wavelength in Figure 4.

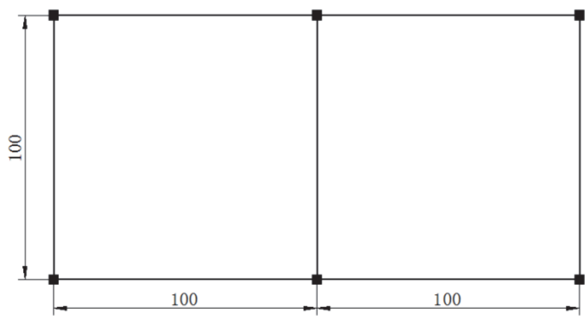


Figure 3: Double-cell box section

Table 2: Mode shapes.

CONCLUSIONS

A new approach to the modal decomposition problem is presented. Transverse extension modes are first separated from the remainder of the deformation space by imposing the null-transverse-stress criterion. Second, the participation of the plate bending energy versus the membrane deformation energy in the total energy is minimized (or maximized) to obtain the local (and global/shear) modes. The distortional modes then follow from orthogonality of the solution. The method is illustrated using an example.

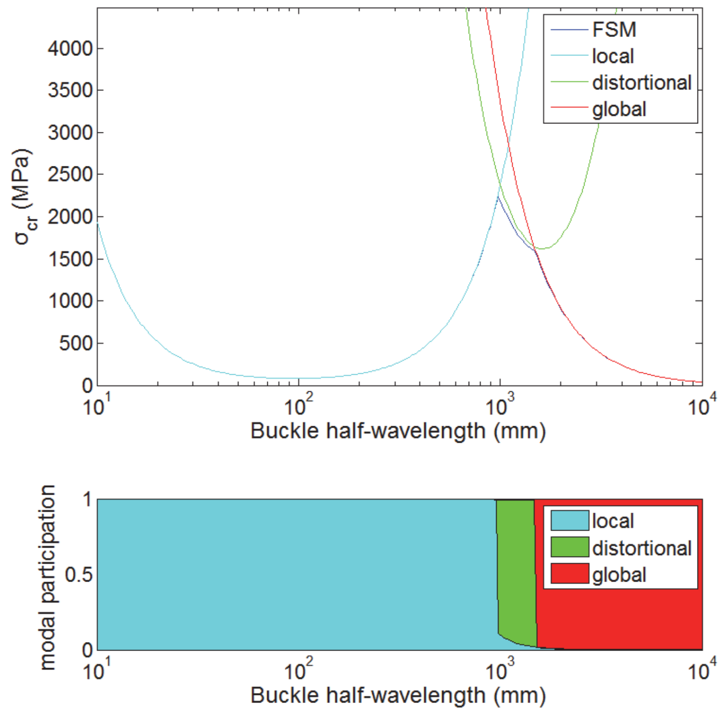


Figure 4: Critical stress and modal participation.

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