NOTCHES AND KINKED CRACKS IN MATERIALS WITH GENERAL ANISOTROPY

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The singular stresses at the tip of an angular notch are analysed for the most general case of elastic anisotropy. The problem is stated in relation with the more general problem of a kinked crack. This is modelled by means of continuous distributions of dislocations which are assumed to be singular at both crack tips and at the kink vertex, the dominant singularity at the kink being weaker than at the crack tips and dependent upon the kink angle and the material properties. A noteworthy result is that the dominant stress field at the notch can be characterised by a single generalised stress intensity factor in the most general cases where there is little or no symmetry. The resulting integral equations are solved numerically with the help of the reciprocal theorem. The stress intensity factors for modes I, II and III and the generalised stress intensity factor at the kink vertex are derived directly from the dislocation densities.

INTRODUCTION

Most attempts to solve the problem of a kinked crack have been for isotropic materials. Very few general solutions are available for anisotropic solids and even so, most analyses are for orthotropic materials. In a neighbourhood of the kinked crack vertex, the stresses are singular like $r^{-\lambda}$ with $0 < \lambda < 1/2$, r being the radial distance to the vertex (Bogy (1)) but this has not always been taken into account either because the singularity has not been considered or because the analyses proceed with a wrong singularity $r^{-1/2}$ at the vertex.

This paper summarises an analysis of the vertex singularity for an arbitrary angle in the most general case of elastic anisotropy. This is used to model the kinked crack as a continuous distribution of dislocations that is singular at both crack tips and at the vertex of the kinked crack. The resulting integral equations are solved numerically by the Chebyshev

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polynomials technique and applying the reciprocal theorem. The stress intensity factors (SIFs) at the crack tips and the generalised stress intensity factor (GSIF) at the kink vertex are obtained directly from the resulting dislocation densities. The method is checked against available solutions for some special cases of anisotropy (orthotropic and isotropic materials), which are analysed as particular cases of the general formulation.

ANALYSIS

The geometry of the kinked crack in an elastic material with general anisotropy is shown in Fig. 1: the main crack lies on the x_1-axis, the origin coincides with the vertex and the crack tip with the point $(-1,0)$; the branch of length c forms an angle ϕ with the main crack. The problem has been modelled as a continuous distribution of dislocations that is singular at both crack tips ($r^{-1/2}$) and at the vertex ($r^{-\lambda}$). Although the analysis of the vertex singularity does not require the use of finite crack and branch lengths, this approach is convenient for solving the problem of the kinked crack. Using this formulation combined with the Mellin transform, Blanco et al. (2) have shown that the dominant stress singularity λ at the vertex is given by the root of greatest real part of the characteristic equation

$$\det[\mathbf{M}(s)] = \det\begin{pmatrix} \frac{1}{2}\cot(\pi s)\,\delta_{kj} & -\frac{1}{2}\csc(\pi s)H_{kj}(s-1) \\ \frac{l_{ki}}{2}\csc(\pi s)H_{ij}(1-s) & -\frac{l_{kj}}{2}\cot(\pi s) \end{pmatrix} = 0 \qquad (1)$$

where l_{ij} are the direction cosines of the axes (x,y) with respect to (x_1,x_2) and $H_{ij}(s)$ are functions of the complex variable s which depend on the material constants and the angle ϕ (2). As expected, λ is independent of the branch length taken to develop the formulation. Moreover, as shown in (2), the rank of matrix $\mathbf{M}(\lambda)$ is five for the general case ($0 < \phi < \pi$) and then the corresponding eigenvector $\mathbf{V} = v_j$ ($j = 1,2,\ldots,6$) is determined up to a multiplicative constant, A, that represents a generalised stress intensity factor (GSIF) at the notch tip. However in the limit case of a crack ($\phi = \pi$) the rank is 3 with $\lambda = 1/2$.

The crack path, see Fig. 1, can be represented as

$$X_1(r) = \begin{cases} r & -1 \le r \le 0 \\ r\cos\phi & 0 < r \le c \end{cases} \quad \text{and} \quad X_2(r) = \begin{cases} 0 & -1 \le r \le 0 \\ r\sin\phi & 0 < r \le c \end{cases} \qquad (2)$$

The dislocation densities $d_j(r)$ are assumed to have square-root singularities at the ends of the interval $[-1, c]$ and a weaker singularity at $r = 0$, and then

$$d_j(r) = \begin{cases} (1+r)^{-1/2}(c-r)^{-1/2} D_j(r) + A v_j (-r)^{-\lambda} & (-1 < r < 0) \\ (1+r)^{-1/2}(c-r)^{-1/2} D_j(r) + A v_{j+3} r^{-\lambda} & (0 < r < c) \end{cases} \qquad (3)$$

where $D_j(r)$ are unknown bounded functions in $[-1, c]$, and A is an unknown constant. The stresses generated by these dislocation densities can be calculated by superposition of the solution for an isolated dislocation due to Stroh (3). Let $T_i(r)$ denote the tractions induced on the crack surface by the external load in the non-cracked medium. The condition that the crack is traction free requires

$$-T_i(r) = \frac{1}{4\pi}\sum_{\alpha} L_{i\alpha} M_{\alpha j}\Omega_\alpha(r)\int_{-1}^{c}\frac{(1+\rho)^{-1/2}(c-\rho)^{-1/2}D_j(\rho)}{X_1(r)-X_1(\rho)+p_\alpha(X_2(r)-X_2(\rho))}\mathrm{d}\rho$$
$$+\frac{A}{4\pi}\sum_{\alpha} L_{i\alpha} M_{\alpha j}\Omega_\alpha(r)\left\{v_j\int_{-1}^{0}\frac{(-\rho)^{-\lambda}}{X_1(r)+p_\alpha X_2(r)-\rho}\mathrm{d}\rho\right. \tag{4a}$$
$$\left.+v_{j+3}\int_{0}^{c}\frac{\rho^{-\lambda}}{X_1(r)+p_\alpha X_2(r)-\rho\tau_\alpha}\mathrm{d}\rho\right\}+C.C. \qquad (-1<r<c)$$

$$\Omega_\alpha(r) = \frac{\mathrm{d}X_1}{\mathrm{d}r}(r) + p_\alpha\frac{\mathrm{d}X_2}{\mathrm{d}r}(r) \tag{4b}$$

$$\tau_\alpha = \cos\phi + p_\alpha\sin\phi \tag{4c}$$

where C.C. denotes the complex conjugate and the convention of summing over repeated Latin indices is used; p_α, $L_{i\alpha}$ and $M_{\alpha j}$ are material dependent constants designated with the same name in Stroh's work (3). The condition that the crack is closed at both ends gives

$$\int_{-1}^{c}(1+\rho)^{-1/2}(c-\rho)^{-1/2}D_j(\rho)\mathrm{d}\rho + A\left[v_j\int_{-1}^{0}(-\rho)^{-\lambda}\mathrm{d}\rho + v_{j+3}\int_{0}^{c}\rho^{-\lambda}\mathrm{d}\rho\right] = 0 \tag{5}$$

The numerical solution of the system of integral equations (4a) and (5) can be obtained by expanding the unknown functions $D_j(\rho)$ in the form

$$D_j(\zeta) = \sum_{n=0}^{N} D_{jn}T_n(\zeta) \tag{6}$$

where T_n are Chebyshev polynomials of the first kind, D_{jn} are unknown coefficients and ζ is a normalised co-ordinate over he interval [-1,1]. Substituting (6) into (4a) and (5) and using the integration properties of the polynomials T_n yield a system of $3N + 3$ equations (Blanco et al. (4)) and $3N + 4$ unknowns: the constant A and the coefficients D_{jn} (j = 0,1,...,N). A supplementary equation can be obtained using the reciprocal theorem for two dimensional elastostatics in the absence of body forces

$$\int_C (\sigma_k\hat{u}_k - \hat{\sigma}_k u_k)\mathrm{d}l = 0\text{, with } \sigma_k = \sigma_{kj}n_j \text{ and } \hat{\sigma}_k = \hat{\sigma}_{kj}n_j \tag{7}$$

where n_j is the outward pointing normal to the simple closed contour C containing no singularities, and $\hat{\sigma}_{kj}$ and $\hat{u}_k$ are two auxiliary fields of stress and displacement which satisfy the same equilibrium and constitutive equations as the actual fields. The appropriate auxiliary fields derived in (2) are originated by a dislocation density given by

$$\hat{d}_j(r) = \begin{cases} \dfrac{w_j}{(-r)^{2-\lambda}} & (-\infty < r < 0) \\ \dfrac{w_{j+3}}{r^{2-\lambda}} & (0 < r < \infty) \end{cases} \tag{8}$$

where $\mathbf{W} = w_i$ $(i = 1,2,\ldots,6)$ is the eigenvector of matrix $\mathbf{M}$ corresponding to the eigenvalue $s = 2-\lambda$. The stress and displacement fields created by these dislocation densities can be calculated, as before, by superposition of the solution for an isolated dislocation. Now consider a closed contour as the one shown in Fig. 1. If Γ_ε and Γ_R are chosen as circular paths of radius $\varepsilon \to 0$ and $R \to \infty$, respectively, it can be proved that the integral along Γ_R vanishes and then the reciprocal theorem becomes

$$A(C_1 - C_2) = \int_{L_1} \sigma_k \hat{u}_k \, \mathrm{d}l + \int_{L_2} \sigma_k \hat{u}_k \, \mathrm{d}l \tag{9}$$

where C_1 and C_2 are constants dependent on the elastic properties and the kink angle ϕ (2). Equation (9) provides the additional equation for the $3N + 4$ unknowns.

The SIFs are obtained directly from the dislocation densities. As the stresses are singular near the crack tips, the asymptotic behaviour is governed by the values of the functions $D_j(r)$ at $r = -1$ and $r = c$ for the main crack and for the branch, respectively, or alternatively, at $\zeta = -1$ and $\zeta = 1$, in the normalised co-ordinates. Then

$$K_{\eta(k)} = \begin{cases} -\dfrac{1}{2}\sqrt{\pi\dfrac{1+c}{2}}D_k(-1) = -\dfrac{1}{2}\sqrt{\pi\dfrac{1+c}{2}}\displaystyle\sum_{n=0}^{N}(-1)^n D_{kn} & \text{for the main crack} \\ \dfrac{l_{ki}}{2}\sqrt{\pi\dfrac{1+c}{2}}D_i(1) = \dfrac{l_{ki}}{2}\sqrt{\pi\dfrac{1+c}{2}}\displaystyle\sum_{n=0}^{N}D_{in} & \text{for the branch} \end{cases} \tag{10}$$

with $\eta(1) = 2$, $\eta(2) = 1$ and $\eta(3) = 3$. The generalised stress intensity factor (GSIF), A, at the kink is obtained directly in the resolution of the system of equations.

RESULTS

The analysis of the stress singularity at the notch tip has been checked against a number of available solutions for particular cases of anisotropy. First, the isotropic material can be treated with the present formulation as a limit case of slight anisotropy. The comparison with the known solutions (Atkinson et al. (5); Sih and Ho (6)) results in values of λ with an accuracy of four significant figures. Another check has been done for the particular case of an orthotropic material. For a notch of monocrystalline silicon with an opening angle of 70.52° (ϕ = 109.48°), Heinzelmann et al. (7) give as the maximum singular stress exponent $\lambda = 0.4814$, while the value obtained through our formulation is $\lambda = 0.4806$. In all the cases, the correct limit values of $\lambda = 0$ (for $\phi = 0$) and $\lambda = 0.5$ (for ϕ = 180°) have been obtained.

As an example of the kinked crack problem, Figs. 2 and 3 show the effect of the branch length and branch angle on the GSIF and the SIFs (for brevity, only K_I is given) for a crack lying in a monoclinic material (ethylene diamine tartrate). The elastic constants for this material have been taken from Huntington (8). The SIFs are normalised with respect to the SIF of the straight crack before branching takes place, $K_0 = \sigma\sqrt{\pi b/2}$, where σ is the stress normal to the main crack and b is the length of the main crack. As expected, for small values of c/b, K_I at the main crack tip tends to K_0. The GSIF is normalised by $\sigma\sqrt{\pi}$ (note that the units of the GSIF ($ML^{1+\lambda}T^{-2}$) depend on the magnitude of the stress exponent λ). The GSIF changes sharply when the branch length is small and tends to stabilise towards the stationary value corresponding to an infinite notch as the branch length increases.

CONCLUSIONS

A method has been developed to obtain the stress singularity at the tip of an angular notch in materials with general anisotropy. This is used to formulate a model for a kinked crack which incorporates the accurate analysis of the main singularity at the kink vertex. The validity of the analysis has been checked against available solutions for isotropic and orthotropic materials, which have been analysed as particular cases of the formulation. Results have been presented that show the influence of the branch length and branch angle on the SIFs and GSIF when the case of general anisotropy is considered.

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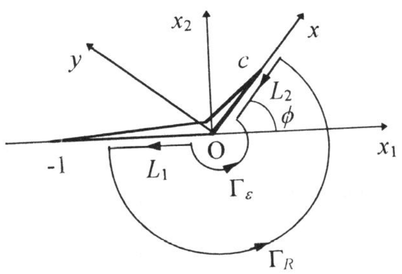

Figure 1. Kinked crack and contour for application of the reciprocal theorem.

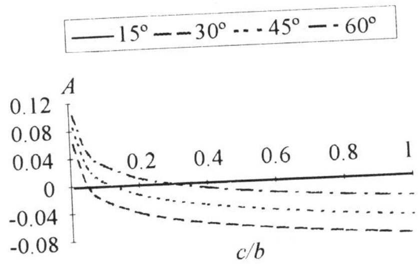

Figure 2. Generalised stress intensity factor as a function of branch length and angle.

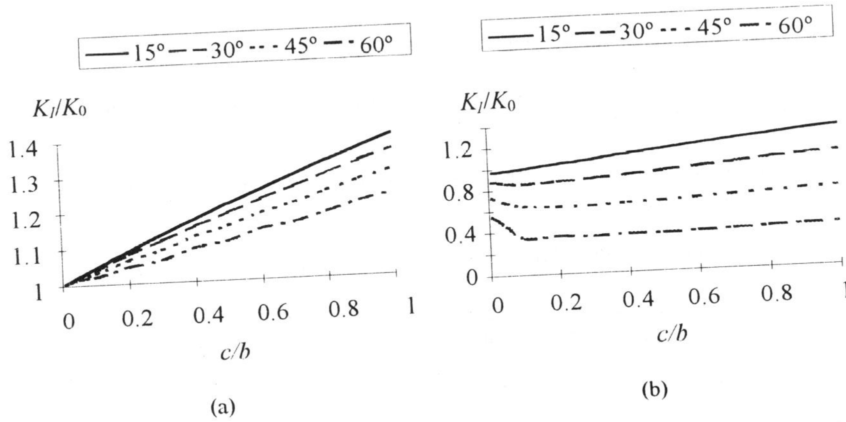

Figure 3. Mode I stress intensity factor as a function of the branch length and branch angle: (a) main crack tip; (b) branch tip.