NUMERICAL METHOD FOR DETERMINATION OF THE MIXED MODE CRACK BEHAVIOUR IN ELASTIC-PLASTIC MATERIALS

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In this paper, a numerical method was proposed to estimate the plastic mixity parameter M^p for an elastic-plastic plane strain crack under mixed mode loading. An associated J-integral, the J^*-integral, has been defined. This integral is path-independent and can be evaluated from two sets of elastic-plastic fields, one being the actual field and another the auxiliary field. The last one can be obtained by decomposition of the actual field into symmetrical and anti-symmetrical parts with respect to the crack axis. By examining the J^*-integral in the near-tip fully plastic zone, the relationship between the J^*-integral and the plastic mixity parameter has been established. This method allows a simple evaluation of the plastic mixity parameter without considering the near tip stress field. The numerical studies show that the present method is quite accurate in the determination of the mode mixity parameter comparing with the finite-element results.

INTRODUCTION

In plane elastic-plastic mechanics, few methods have been developed to separate the mixed modes. Shih [1] generalized the solution of Hutchinson [2] and Rice and Rosengren [3] (the HRR solution), initially derived for mode I plastic cracks, for corresponding mixed mode problems. He showed that two parameters, the J-integral [4] and the plastic mixity parameter M^p, define completely the near-tip asymptotic stress field. The mixity parameter M^p was related to the stress intensity factors K_I and K_{II} of the far elastic field by using the finite-element analysis for small-scale yielding. Kishimoto et *al.* [5] developed a path-independent integral, the $\hat{J}$-integral, which can be decomposed into $\hat{J}^I$ and $\hat{J}^{II}$, corresponding respectively the symmetrical and the anti-symmetrical parts of the $\hat{J}$ integral with respect to the crack axis. For linear elastic materials, the $\hat{J}$-integral coincides with the J-integral and $\hat{J}^I$ and $\hat{J}^{II}$ are related respectively to the stress intensity factors K_I and K_{II}. Tohgo and Ishii [6] proposed a simple method to estimate the symmetrical and anti-symmetrical parts of J-integral for single-edge-cracked specimens subjected to bending moment and shearing force. However, the decompositions in [5] and [6] did not lead to determine the plastic mode mixity parameter. Since at present there is no systematic method

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for determining the near-tip plastic mode-mixity, experimental results (for example Tohgo et to *al.* [7], Aoki et *al.* [8]) are being reported using the ratio of the elastic stress intensity factors (K_I/K_{II}). This may be meaningless because large plastic yielding can take place prior fracture initiation in low strength alloys. Therefore, it is very important to find out an general method to determinate the mixity parameter M^p, an essential parameter allowing the determination of the near-tip asymptotic field.

In this paper, an associated *J*-integral, the *J**-integral, was defined in order to decouple the mixed modes of an elastic-plastic crack. The auxiliary fields, constructed from the symmetrical and anti-symmetrical parts of the actual field, were introduced into the *J**-integral. By completing two independent calculations with the two auxiliary fields, two values of the *J**-integral can be obtained for mixed mode cracks. By introducing a simplification in the calculation, the evaluation of these two values in the near-tip region leads to the determination of M^p. The results obtained in this work agree well with those of Shih [1] for cracks under small-scale yielding conditions. Because of the path-independence of the *J**-integral, no analysis of the stress field near the crack tip is required to separate the mixed modes. Moreover, the present method is not limited by the small-scale yielding conditions and can be applied to any platic yielding case.

GENERAL CONSIDERATION

Let us consider two sets of plane-strain elastic-plastic fields (u,ε,σ) and ($u^*,\varepsilon^*,\sigma^*$), where u,ε,σ are respectively the displacement vector, the strain tensor and the stress tensor of the actual field, and $u^*,\varepsilon^*,\sigma^*$ are the corresponding quantities of an auxiliary field. The associated *J*-integral, the *J**-integral, can be defined in Cartesian coordinates as follows:

$$J^* = \int_\Gamma \left(w^* n_1 - \sigma_{ij} n_j \frac{\partial u_i^*}{\partial x} \right) ds \qquad (1)$$

where Γ is an arbitrary path around the crack tip, w^* is the associated strain energy density defined as follows:

$$dw^* = \sigma_{ij} d\varepsilon_{ij}^* \qquad (2)$$

From this definition, it is easy to show that the *J**-integral is path-independent. Following the approach of Ishikawa *et al.* [9], one can decompose the actual field into symmetrical and anti-symmetrical parts with respect to the crack axis. So we introduce the fields u^{I*} and u^{II*} as follows:

$$u_i^{*M}(x,y) = \frac{1}{2}\left[u_i(x,y) + (-1)^{i+M} u_i(x,-y) \right] \quad (i=1,2;\ M=\mathrm{I,II}) \qquad (3)$$

According to (1) and (3), two associated *J*-integral can be defined as follows:

$$J^{*M} = \int_\Gamma \left(w^{*M} n_1 - \sigma_{ij} n_j \frac{\partial u_i^{*M}}{\partial x} \right) ds \quad (M=\mathrm{I,II}) \qquad (4)$$

where $$w^{*M} = \int_0^{\varepsilon_{ij}^{*M}} \sigma_{ij} d\varepsilon_{ij}^{*M} \quad (M=\mathrm{I,II}) \qquad (5)$$

Now we consider the case when the radius of the integral tends to zero such that it lies within the zone dominated by the fully plastic singularity. Shih [1] showed that, for a mixed mode crack lying in a power-law hardening material, the stresses, strains and displacements near the crack tip are dominated by the HRR singularity, and can be represented in polar coordinates (r,θ) as follows:

$$\sigma_{ij} = \sigma_0 K r^{-1/(n+1)} \tilde{\sigma}_{ij}(\theta, M^p)$$
$$\varepsilon_{ij} = \frac{\alpha\sigma_0}{E} K^n r^{-n/(n+1)} \tilde{\varepsilon}_{ij}(\theta, M^p) \qquad (6)$$
$$u_i = \frac{\alpha\sigma_0}{E} K^n r^{1/(n+1)} \tilde{u}_i(\theta, M^p)$$

where σ_0 is the yield stress, α may be regarded as a material constant, n is the strain hardening coefficient, E is Young's modulus, K is the plastic stress-intensity factor and M^p is the mixity parameter near the crack tip defined as follows:

$$M^p = \frac{2}{\pi} \tan^{-1} \left| \frac{\tilde{\sigma}_{\theta\theta}(\theta = 0, M^p)}{\tilde{\sigma}_{r\theta}(\theta = 0, M^p)} \right| \qquad (7)$$

The dimensionless functions $\tilde{\sigma}_{ij}, \tilde{\varepsilon}_{ij}, \tilde{u}_i$ depend only on θ and the mixity parameter M^p and can be determined by the numerical method described in [1]. Under these conditions, the integrands in (4) can be evaluated in terms of two unknowns, the amplitude K and the mixity parameter M^p, which are function of the loading level. One can write

$$w^{*M} = \int_0^{\varepsilon_{ij}^{*M}} \sigma_{ij} d\varepsilon_{ij}^{*M} = \int^{M^p, K} \sigma_{ij} \left(\frac{\partial \varepsilon_{ij}^{*M}}{\partial M^p} dM^p + \frac{\partial \varepsilon_{ij}^{*M}}{\partial K} dK \right) \quad (M\text{=I,II}) \qquad (8)$$

The variation of the mixity parameter M^p during a monotonic loading can be observed from some experimental studies (for example, in Tohgo and Ishii [6]). However, this variation is not very important according to the numerical and experimental studies gathered so far. Consequently, one can suppose that this variation contributes little to the construction of w^* and then the term associated to dM^p can be neglected in (8). The influence on the accuracy of this simplification will be discussed latter. For more convenience, the final value of M^p is taken into account in the calculations. With these assumptions, one obtains:

$$J^{*M} = \frac{\alpha\sigma_0^2}{E} K^{n+1} I_n^{*M} \qquad (M\text{=I,II}) \qquad (9)$$

where
$$I_n^{*M} = \int_{-\pi}^{\pi} \left\{ \frac{n}{n+1} \tilde{\sigma}_{ij} \tilde{\varepsilon}_{ij}^{*M} \cos\theta - \left[\sin\theta \left(\tilde{\sigma}_r (\tilde{u}_\theta^{*M} - d\tilde{u}_r^{*M}/d\theta) - \tilde{\sigma}_{r\theta} (\tilde{u}_r^{*M} + d\tilde{u}_\theta^{*M}/d\theta) \right) + \frac{1}{n+1} \cos\theta \left(\tilde{\sigma}_r \tilde{u}_r^{*M} + \sigma_{r\theta} \tilde{u}_\theta^{*M} \right) \right] \right\} d\theta \quad (M\text{=I,II}) \qquad (10)$$

The integral parameters I_n^{*I} and I_n^{*II} are function of the power hardening coefficient n and mixity parameter M^p.

Now let us consider the case of the small-scale yielding problem. If the radius of the closed path is large enough, the far stress field is governed by the first term of Williams' expansion[10]. In this case, one can easily show that for plane strain.

$$J^{*M} = \frac{1-\nu^2}{E} K_M^2 \qquad (M\text{=I,II}) \qquad (11)$$

where K_I and K_{II} are respectively the stress-intensity factors for purely mode I and mode II cracks. The mixity parameter of the elastic far field M^e was introduced by Shih [1]:

$$M^e = \frac{2}{\pi}\tan^{-1}\left|\lim_{r\to\infty}\frac{\sigma_{\theta\theta}(\theta=0)}{\sigma_{r\theta}(\theta=0)}\right| = \frac{2}{\pi}\tan^{-1}\left|\frac{K_I}{K_{II}}\right| \qquad (12)$$

In the case of the small-scale yielding, it is evident that

$$M^e = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{J^{*I}}{J^{*II}}} \qquad (13)$$

For more general cases when the small-scale yielding condition is not satisfied, it is useful to defined an equivalent elastic mixity parameter M^{*e}:

$$M^{*e} = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{J^{*I}}{J^{*II}}} \qquad (14)$$

It is clear that M^{*e} equals M^e for small-scale yielding. However, M^{*e} has not the same definition as M^e, therefore, is not equivalent to M^e in general yielding cases. The principal advantage of M^{*e} is that it can be evaluated from two integrals at any closed circuits around the crack tip, while the evaluation of M^e requires to know the far elastic field.
The relationship between M^{*e} and M^p can easily be found out according to equation (14). In fact, by substituting (9) into (14), one can write:

$$M^{*e} = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{I_n^{*I}(M^p)}{I_n^{*II}(M^p)}} \qquad (15)$$

For each mixity parameter M^p, the integration parameters I_n^{*I} and I_n^{*II} can be calculated from equation (10). According to (15), the relationship M^{*e}- M^p is established and shown in Fig. 1 for different hardening coefficient n. The determination of the mixity parameter M^p is then easy: the equivalent elastic mixity parameter M^{*e} can first be calculated by means of J^{*I} and J^{*II} according to equation (14), then the plastic mixity parameter M^p can be obtained from Fig. 1.

NUMERICAL VERIFICATIONS AND DISCUSSIONS

In order to verify the path-independence of the J^*-integral and the accuracy of the present method in estimating the mixity parameter M^p, we have carried out a numerical study by using the finite-element modelling. The structure studied is a plane strain cracked beam subjected to a concentrated force P. The geometry of the structure is shown in Fig. 2.

The path-independence: The path-independence of J^* is verified by using several integrating paths with different radius. It was confirmed that J^{*I} and J^{*II} are path-independent.

The accuracy of the estimation of M^p: In this work, four mixed mode loads, d/L= 0.0253, 0.0505, 0.125 and 1, are studied for both materials n=3 and 9. For each load, the equivalent elastic mixity parameter M^{e*} is calculated according to (14), then the plastic mixity parameter M^p can be found out from Fig.1. The possible errors introduced in this approach are related to neglecting the variation with respect to M^p in the evaluation of w^{*I} and w^{*II} cf. equation (8). In order to evaluate these errors, we calculate M^p from the local stresses near the crack tip according to the definition of (7). The stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ at the crack axis ahead the crack tip are calculated by the finite-element analysis. Fig. 3 shows the comparison betwwen the values of M^p obtained from the J^*-integral and those obtained from the local stresses for n=3 and n=9. This figure demonstrates that the present method is quite accurate in the estimation of the plastic mixity parameter M^p.

CONCLUDING REMARKS

In this paper, we have presented a numerical method to estimate the plastic mixity parameter M^p for an elastic-plastic plane strain crack under mixed mode loading. A path-independent integral, the J^*-integral, has been defined by using two sets of elastic-plastic fields, the actual field and the auxiliary fields. The auxiliary fields can be obtained by decomposition of the actual field into symmetrical and anti-symmetrical parts with respect to the crack axis. With some simplifications in the calculation, this method permits a simple evaluation of the plastic mixity parameter M^p without considering the near-tip fields. The numerical studies show that the errors due to the introduced simplifications are not important. The results obtained by using the present method are quite accurate comparing with the finite-elements results. This method is not limited by the small-scale yielding conditions and can be applied to more general yielding cases.

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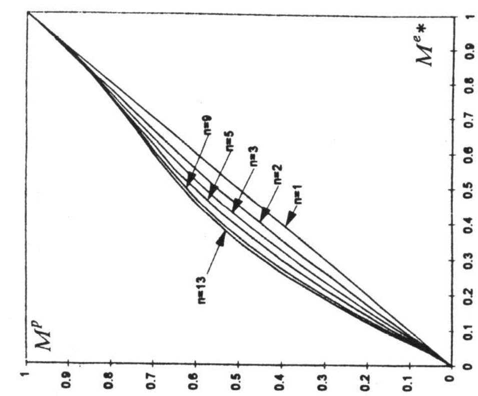

Fig. 1. Plastic mixity M^p versus equivalent elastic mixity M^{e*} in plane strain

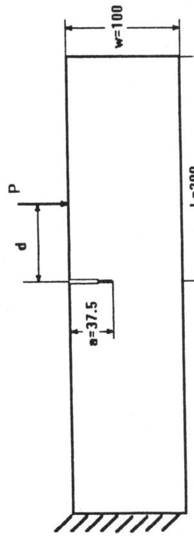

Fig. 2. Geometry of the studied structure

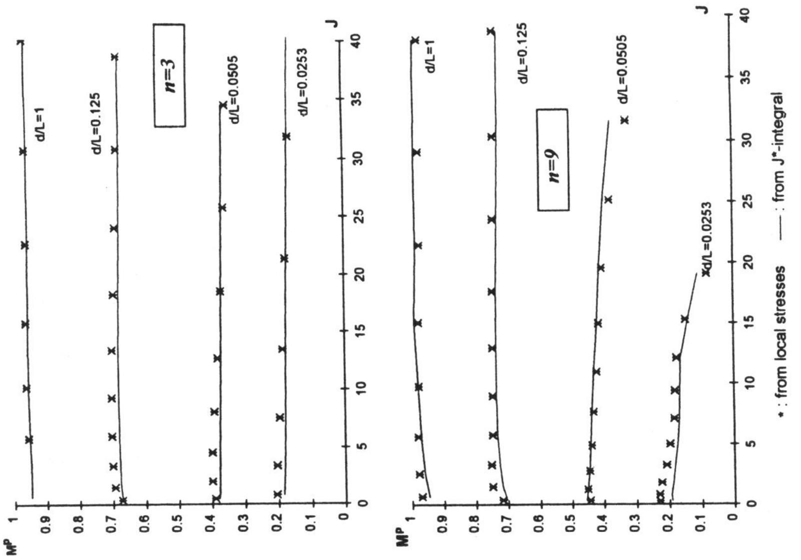

Fig. 3. Variation of M^p during the loading