SOME CRITICAL REMARKS ON THE DUGDALE STRIP YIELD MODEL FOR THE CRACK TIP PLASTICITY

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The mechanical response of an infinite thin plate with a central straight crack is considered in this paper. On the plate boundaries at infinity the biaxial normal in-plane tractions are applied, while the crack surface is loading free. It is assumed that the magnitude of the applied loads will cause the evolution of the plastic deformations in the neighbourhood of the crack tips. The plasticity aspect of fatigue crack growth will be of interest to this article. The generalised Dugdale-Barenblatt strip yield model is utilised for a description of the plastification process. In this paper two yield criteria are considered, the Tresca and the Mises, respectively. The analysis of the investigated mechanical response, which includes stresses, displacements, stress intensity factors, plastic zone magnitude, crack tip opening displacement, residual stress distribution etc., is carried out analytically using methods of the theory of analytical functions of a complex variable.

DUGDALE'S MODELLING OF COHESIVE ZONE FOR THE DUCTILE FRACTURE

The crack tip plasticity analysis can be performed according to the premises of the Dugdale-Barenblatt strip yield model, Rice (1). In this investigation the Dugdale strip yield model which is in fact a simplification of the more complex Barenblatt yield model will be assumed for modelling of the cohesive zone for the ductile fracture. When posing his hypothetical model, Dugdale assumed first that plastic deformation is governed by perfect plasticity under the supposition of constant cohesive stress $\sigma_{yy} = \sigma_0$ in the yielded area, with σ_0 being the yield strength, and second that yielding of a material is confined to a narrow strip band, extending ahead from the crack tip and lying along the crack direction. Accordingly, he postulated the existence of an imaginary elastic crack composed of a physical blunt crack of length $2a$ and a supplementary cracked zone extended ahead at both tips of the virgin sharp crack for a distance r_p, the length of the supplementary crack being equal to the length of the cohesive zone around the crack tip.

The determination of stresses and displacements in the yielded plate is obtained as a superposition of two elastic responses, both taking the imaginary crack of length $2b$ into account. Actually, the elastic response due to the external loading of the modified cracked plate is superposed by the elastic response due to the application of the cohesive stresses.

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Because of the assumed elastic approach both responses are characterised by the stress singularity, their intensities being given by the stress intensity factors K_{ext} and K_{coh} respectively. But, since in reality the stress singularity, introduced by the elastic approach, does not occur due to plastic yielding it has to be cancelled by imposing

$$K(a+r_p) = K_{ext}(a+r_p) + K_{coh}(a+r_p) = 0 \qquad (1)$$

The fulfilment of the above condition yields the plastic zone length r_p.

In our present investigation the theoretical framework to deal with the fatigue crack growth in a residual stress field is also established. A detailed analysis of the plasticity around a fatigue crack is performed based on the strip yield assumption. According to the strip yielding assumption, the requirement which removes the stress singularity at the plastic zone boundary in the residual stress field becomes

$$K(a+r_p) = \left[K_{ext}(a+r_p) + K_{res}(a+r_p)\right] + K_{coh}(a+r_p) = 0 \qquad (2)$$

where $K_{res}(a+r_p)$ is the K factor due to the residual stress field.

ELASTIC-PLASTIC BOUNDARY VALUE PROBLEM DEFINITION

An infinite plate $(z \in D,\ D\!: |z| \geq 0)$ made of the ductile material with an embedded straight crack of length $2a$ lying on the x-axis $(z \in L,\ L\!: \mathrm{Re}|z| \leq a,\ \mathrm{Im}\, z = 0)$, its material being supposed to exhibit elastic-perfectly plastic behaviour or elastic linear strain-hardening properties, is considered. An in-plane remote loading, static or cyclic, is assumed to be applied symmetrically in respect of the x- and y-axes, while the crack boundary is traction free. By adopting Dugdale's approach for the crack tip plasticity the original elastic-plastic boundary value problem that is defined on the domain D cut along the line L has to be adequately modified. The modified problem is an elastic one and is defined on the domain $D^*(z \in D^*,\ D^*\!: |z| \geq 0)$ cut along the line $L^*(z \in L^*,\ L^*\!: \mathrm{Re}|z| \leq b = a + r_p,\ \mathrm{Im}\, z = 0)$.

In this investigation we are particularly interested in the remote loading which is biaxially dependent, the boundary conditions at infinity are defined on the following manner

$$\sigma_{xx}(z) = \sigma_{xx}^{\infty} = k\sigma_{\infty},\quad \sigma_{yy}(z) = \sigma_{yy}^{\infty} = \sigma_{\infty},\quad \sigma_{xy}(z) = \sigma_{xy}^{\infty} = 0 \qquad as\ |z| \to \infty \quad (3)$$

where $\sigma_{\infty}\ (\sigma_{\infty} > 0)$ and k are real constants. On the edges of the imaginary crack, i.e. on the line L^*, the correspondent boundary conditions are as follows

$$\begin{aligned} &\sigma_{yy}(z) = \sigma_{xy}(z) = 0 && \text{for } z \in L, \\ &\sigma_{yy}(z) = \sigma_{Y}(z),\quad \sigma_{xy}(z) = 0 && \text{for } z \in L^* - L, \\ &\sigma_{yy}(z) = \sigma_{res}(z),\quad \sigma_{xy}(z) = 0 && \text{for } z \in L^*. \end{aligned} \qquad (4)$$

Here, symbol $\sigma_Y(z)$ is introduced to denote the variable or constant cohesive stress along the yielded zone, its magnitude not being necessary equal to the yield strength σ_0. On the other hand symbol $\sigma_{res}(z)$ denotes the residual stress distribution along part of an imaginary elastic crack.

BASIC EQUATIONS EXPRESSED IN TERMS OF THE COMPLEX VARIABLE THEORY

Basic equations of the modified problem, which can be considered as a plane stress problem, are those of the linear theory of elasticity in conjunction with any yield criterion. The yield criterion governing the evolution of plastic yielding and implicitly affecting the magnitude of the cohesive stress σ_Y. Solution of the above defined problem can be achieved with different methods. In order to obtain analytical solution we have adopted the methodology based on Muskhelishvili's complex variable theory of elasticity, Muskhelishvili (2). Introducing two complex potentials $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x + iy$, Kolosov has derived the expressions for the complex representation of the stress and displacement fields for the general plane stress problem.

Considering the fact that the applied remote loading are symmetric with respect to the x-z plane, the shear stress σ_{xy} vanishes at all points along the x-axis, $\operatorname{Im} z = 0$. The governing equations of the plane theory of elasticity can be expressed in terms of one single Westergaard function $Z(z)$, as

$$\sigma_{xx} + \sigma_{yy} = Z(z) + \overline{Z(z)}, \qquad \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2A - (z - \bar{z})Z'(z) \tag{5}$$

$$2\mu(u + iv) = \frac{1}{2}\left[\frac{3-\nu}{1+\nu}\int Z(z)\,\mathrm{d}\,z - \int \overline{Z(z)}\,\mathrm{d}\,\bar{z} - (z - \bar{z})\overline{Z(z)}\right] - A\bar{z}$$

where A is a real constant, μ is the shear modulus and ν is Poisson's ratio.

TRESCA AND MISES YIELD CRITERION IMPLEMENTATION IN THE COHESIVE MODEL FOR THE CRACK TIP PLASTICITY

In our further investigation of the crack tip plasticity we consider two yield criteria, the Tresca criterion and the Mises criterion. The assumed strip yield model depends principally on the evolution of plastic range along the x-axis or along the crack direction. The symmetry condition yields that the shear stress $\sigma_{xy} = 0$ at all points along the x-axis and consequently $\sigma_{xx}(z)$ and $\sigma_{yy}(z)$ are two principal stresses for $\operatorname{Im} z = 0$. The third principal stress is zero for plane stress. So, the two yield criteria can be written in terms of principal stresses as follows

$$\left[(\sigma_{xx}(z) - \sigma_{yy}(z))^2 - \sigma_0^2\right]\left[\sigma_{xx}^2(z) - \sigma_0^2\right]\left[\sigma_{yy}^2(z) - \sigma_0^2\right] = 0, \qquad \text{for } \operatorname{Im} z = 0 \tag{6}$$

for the Tresca yield criterion, and

$$\sigma_{xx}^2(z) - \sigma_{xx}(z)\sigma_{yy}(z) + \sigma_{yy}^2(z) = \sigma_0^2, \qquad \text{for } \operatorname{Im} z = 0 \tag{7}$$

for the Mises yield criterion, respectively.

The magnitude of the constant cohesive stress $\sigma_{yy} = \sigma_Y$ in the cohesive zone to be used in our computations is to be consistent with the actual stress state along the x-axis and any of two yield criteria considered. Consistency of stresses along the x-axis implies that the difference of the normal stresses is constant and proportional to the applied loading at infinity

$$\sigma_{yy}(z)-\sigma_{xx}(z)=2A=(1-k)\sigma_{\infty}, \qquad \text{for } \operatorname{Im} z=0 \quad (8)$$

Combination of equations (7) and (8) yields

$$\sigma_{yy}^{2}(z)-(1-k)\sigma_{\infty}\sigma_{yy}(z)+\left[(1-k)^{2}\sigma_{\infty}^{2}-\sigma_{0}^{2}\right]=0 \qquad \text{for } \operatorname{Im} z=0 \quad (9)$$

from which it is immediately obtained the magnitude of the constant cohesive stress $\sigma_{yy}(z)=\sigma_{Y}(z)$ in the cohesive zone based on the Mises yield criterion

$$\sigma_{yy}^{Mises}=\sigma_{Y}^{Mises}=\sigma_{0}\left[\sqrt{1-\frac{3}{4}(1-k)^{2}\left(\frac{\sigma_{\infty}}{\sigma_{0}}\right)^{2}}+\frac{1-k}{2}\frac{\sigma_{\infty}}{\sigma_{0}}\right] \qquad (10)$$

Now we return to the Tresca yield criterion (6). It can be demonstrated, Štok (4), that consistency of the crack tip zone yielding is proper only if

$$\sigma_{yy}^{Tresca}=\sigma_{Y}^{Tresca}=\begin{cases}\sigma_{0} & \text{for } k\le 1\\ \sigma_{0}\left[1+(1-k)\dfrac{\sigma_{\infty}}{\sigma_{0}}\right] & \text{for } k>1\end{cases} \qquad (11)$$

is taken for the cohesive stress σ_Y. Some restrictions regarded to the magnitude of the applied loading and the biaxial load ratio k have been discussed in the papers (3) and (4). The range of admissible values for the biaxial load ratio k is obtained by considering a stable solution with the plastic zone localised at the crack tips and positive cohesive stress σ_Y. It is important to emphasise the role of the biaxial load ratio k in the evolution of plastic yielding in the cohesive zone. Namely, nevertheless that the biaxial load ratio k is assumed fixed for a considered loading case the cohesive stress σ_Y is subject to variation from the yield strength σ_0 to a minimum value, accordingly to the gradual application of the remote loading σ_∞. The variation of the magnitude of the cohesive stress σ_Y with a monotonous increasing of the applied loading σ_∞ and the biaxial load ratio k in dependence upon the applied yield criterion is represented on Figs 1-2 for positive values of the factor k.

REVIEW OF RESULTS

Stresses and displacements in the domain D can be readily determined by considering relationships (5), while in order to characterize the fracture behaviour it is convenient to determine the crack tip opening displacement δ_t and the plastic extension ahead of the crack tip r_p. Both parameters can be expressed explicitly in terms of the applied remote loading σ_∞ and the correspondent cohesive stress σ_Y

$$\frac{\delta_t}{a}=\frac{8}{\pi}\frac{\sigma_Y}{E}\ln\frac{b}{a}=\frac{8}{\pi}\frac{\sigma_Y}{E}\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma_{yy}^{\infty}}{\sigma_Y}\right)\right] \quad (12), \qquad \frac{r_p}{a}=\frac{b}{a}-1=\sec\left(\frac{\pi}{2}\frac{\sigma_{yy}^{\infty}}{\sigma_Y}\right)-1, \quad (13)$$

SOME ASPECTS OF FATIGUE CRACK GROWTH IN RESIDUAL STRESS FIELDS

In this section, the effect of residual stress on fatigue crack growth is considered. If we involve a residual stress field in the investigation, the cyclic loading problem treatment is the same as for the static fracture problem. Elastic-plastic stress, strain and displacement fields ahead of the crack tip can be obtained by the plastic superposition method developed by Rice. This method is attractive and widely used due to its simplicity.

Under small scale yielding conditions, the crack tip opening displacement δ_{t1} and the monotony plastic yield zone size r_{p1} before unloading i.e. at the maximum external load level $\sigma_{\infty 1}$, can be computed according to the expressions (12) and (13) as

$$\delta_{t1} \equiv v^{+}(a,0) - v^{-}(a,0) = \frac{8}{\pi}\frac{\sigma_{Y1}}{E} a \ \ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma_{\infty 1}}{\sigma_{Y1}}\right)\right] = \frac{8}{\pi}\frac{\sigma_{Y1}}{E} a \ \ln\left(\frac{r_{p1}}{a} + 1\right) \quad (14)$$

$$\frac{r_{p1}}{a} = \frac{b_1}{a} - 1 = \sec\left(\frac{\pi}{2}\frac{\sigma_{\infty 1}}{\sigma_{Y1}}\right) - 1 \quad (15)$$

Similarly, the crack tip closing displacement δ_{try} and the magnitude of the crack tip plastic zone r_{pry} in the process of reversed yielding i.e. at the compressive stress level $\Delta\sigma_{\infty 1}$ applied perpendicular to the crack direction amount

$$\delta_{try} = \frac{8}{\pi}\frac{\sigma_{Yry}}{E} a \ \ln\left[\sec\left(\frac{\pi}{2}\frac{\Delta\sigma_{\infty 1}}{\sigma_{Yry}}\right)\right] = \frac{8}{\pi}\frac{\sigma_{Yry}}{E} a \ \ln\left(\frac{r_{pry}}{a} + 1\right) \quad (16)$$

$$\frac{r_{pry}}{a} = \frac{b_{ry}}{a} - 1 = \sec\left(\frac{\pi}{2}\frac{\Delta\sigma_{\infty 1}}{\sigma_{Yry}}\right) - 1 \quad (17)$$

Finally, in accordance with the plastic superposition method developed by Rice the final crack tip opening displacement δ_{tr} (r - residual) and the residual crack tip plastic zone size r_{pr} after unloading, i. e. after reducing the external load level $\sigma_{\infty 1}$ to $\sigma_{\infty 1} - \Delta\sigma_{\infty 1}$, can be computed according to the superposition method by subtracting the correspondent expressions (14) and (16) as well as (15) and (17), respectively

$$\delta_{tr} = \delta_{t1} - \delta_{try} = \frac{8}{\pi}\frac{\sigma_{Y1}}{E} a \ \ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma_{\infty 1}}{\sigma_{Y1}}\right)\right] - \frac{8}{\pi}\frac{\sigma_{Yry}}{E} a \ \ln\left[\sec\left(\frac{\pi}{2}\frac{\Delta\sigma_{\infty 1}}{\sigma_{Yry}}\right)\right] = \\ = \frac{8}{\pi}\frac{\sigma_{Y1}}{E} a \ \ln\left(\frac{r_{p1}}{a} + 1\right) - \frac{8}{\pi}\frac{\sigma_{Yry}}{E} a \ \ln\left(\frac{r_{pry}}{a} + 1\right) \quad (18)$$

$$r_{pr} = r_{p1} - r_{pry} = a \ \sec\left(\frac{\pi}{2}\frac{\sigma_{\infty 1}}{\sigma_{Y1}}\right) - a \ \sec\left(\frac{\pi}{2}\frac{\Delta\sigma_{\infty 1}}{\sigma_{Yry}}\right) \quad (19)$$

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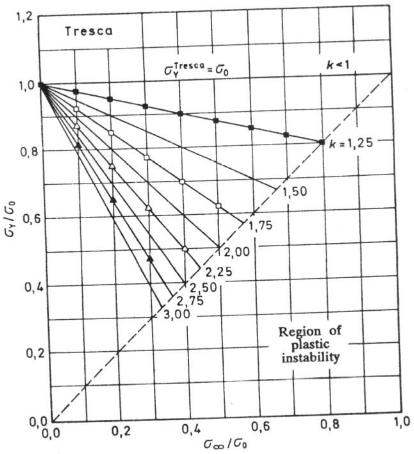

Figure 1. Dependence of the cohesive stress σ_Y upon the biaxial load ratio k;, Tresca

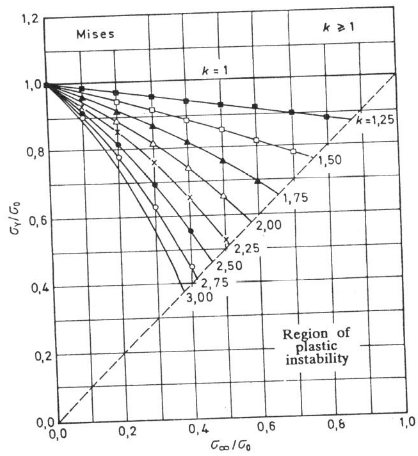

Figure 2. Dependence of the cohesive stress σ_Y upon the biaxial load ratio k,;Misses.