LOCALISATION OF DEFORMATION VIEWED AS A LOSS OF QUASI-STATICITY

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The localisation of deformation is modelled as a local loss of quasi-staticity for an equilibrium problem. Considering rate-dependent materials for which the reversible part of the strain is rate-independent, a criterion for the onset of the localisation bifurcation is built. Its construction is based on the assumption that local dynamic phenomena trigger localisation. The latter is predicted when both the condition for a bifurcation of an auxiliary local problem and a critical value of the inertia terms are simultaneously attained.

INTRODUCTION

The localisation of deformation in solids is a material instability usually modelled as a bifurcation from a homogeneous state to an heterogeneous one (Rice [1]). The emergence of a localisation band is viewed as the onset of a singular surface (limiting the band) for the strain rate. The compatibility conditions, linked to this discontinuity, combined with the constitutive rate-independent laws lead, while considering plastic models, to the well known condition $\det(\underline{n}.\tilde{\tilde{D}}.\underline{n})=0$, where $\underline{n}$ is the unit normal to the singular surface and $\tilde{\tilde{D}}$ is the dissipative branch of the fourth order tangent stiffness tensor. However, when applying this modelling (valid for rate-independent materials) to rate-dependent solids, one finds the criterion $\det(\underline{n}.\tilde{\tilde{E}}.\underline{n})=0$ where $\tilde{\tilde{E}}$ is the elastic stiffness tensor; $\tilde{\tilde{E}}$ being positive definite, this condition is never satisfied. As localisation is observed in viscid materials, there is a paradox that shall be lifted.

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Besides, the post-localisation behaviour of materials is often characterised by either the emergence of a macroscopic crack leading to fracture (brittle materials like rocks and concretes) or a softening regime ending also by fracture events (ductile materials like steels). In both cases the strains concentrate quickly in the localisation band and this rapid evolution leads us to consider that the localisation of deformation marks a beginning of dynamic phenomena whatever the loading (quasistatic or dynamic). The problem of localisation bifurcation in a large spectrum of inelastic solids viewed as an incipient dynamic problem, is the subject of this paper.

To argue this point. a large class of rate dependent materials and its associated boundary-value problem (P) are first presented. Then, by performing a linear perturbation stability analysis, a local auxiliary problem $(\ddot{P})$, which is (P) derivated twice with respect to time, is proved to bifurcate. Is given afterwards, a quantitative definition of the quasi-staticity threshold namely a critical value of the norm of the inertia terms in the linear momentum balance. It is considered that if the norm is satisfied, the material behaviour is locally dynamic. By linking the two latter points, i.e. by considering the bifurcation of $(\ddot{P})$ corresponding to the the critical value of the inertial effects, a localisation criterion for rate dependent materials is settled. Its verification corresponds to a transition from a (quasi)-static equilibrium to a dynamic one (locally) even for external loadings regarded as quasistatic.

A CLASS OF RATE-DEPENDENT MATERIALS

Constitutive rate equations

Attention is focused hereafter on a large class of constitutive equations, established under the small strain framework and in isothermal conditions (Benallal et al. [6]). We admit the reversible behaviour of these materials to be determined by a free energy $w(\tilde{\epsilon}, \tilde{\alpha})$, function of the strain $\tilde{\epsilon}$ and a given number of internal variables $\tilde{\alpha}$ (scalars, vectors or tensors). Thereafter, the stress tensor $\tilde{\sigma}$ and the thermodynamic forces $\tilde{A}$ connected with the internal variables $\tilde{\alpha}$, are given by the state laws:

$$\tilde{\sigma} = \frac{\partial w}{\partial \tilde{\epsilon}}(\tilde{\epsilon}, \tilde{\alpha}) \qquad \tilde{A} = -\frac{\partial w}{\partial \tilde{\alpha}}(\tilde{\epsilon}, \tilde{\alpha}) \tag{1}$$

As far as the material irreversible behaviour is concerned, we define a convex reversibility domain limited by the criterion f=0 (in the space of forces) where f is a function of $\tilde{A}$, eventually parameterised by $\tilde{\alpha}$, i.e. f($\tilde{A}$;$\tilde{\alpha}$); inside this domain no irreversibility is possible. We also assume the existence of a pseudo-potential g($\tilde{A}$;$\tilde{\alpha}$) from which the evolution laws, assuming $\dot{\tilde{\alpha}}$ normality, follow:

$$\dot{\tilde{\alpha}} = \Lambda \frac{\partial g}{\partial \tilde{A}} \tag{2}$$

where Λ can be called pseudo-visco-non-linear multiplier. The elasto-visco-non-linear class of models considered here is an extension of the elasto-viscoplastic materials first proposed by Perzyna [7]. The multiplier is given by:

$$\Lambda = \frac{1}{\eta} < \frac{f}{K} >^n \tag{3}$$

where <> are the MacCauley brackets, η is a relaxation time, n a dimensionless viscosity exponent and K a resistance coefficient that depends on the material mechanical state, say K=K($\tilde{\epsilon}, \tilde{\alpha}$). By derivating (1) with respect to time, the rate law is obtained:

$$\dot{\tilde{\sigma}} = \tilde{\tilde{E}}^{w} : \dot{\tilde{\epsilon}} + \tilde{B}(\tilde{\epsilon}, \tilde{\alpha}) \tag{4}$$

$\tilde{\tilde{E}}^{w} = \frac{\partial^2 w}{\partial \tilde{\epsilon} \partial \tilde{\epsilon}}$ is the elastic stiffness tensor and $\tilde{B}$ is a function of non rate terms.

Boundary-value problems

Considering a structure Ω subjected to body forces $\underline{f}^g$ and its boundary $\partial\Omega$ subjected to surface tractions $\underline{F}^g$ on the part $\partial\Omega_1$, and to given displacements $\underline{u}^g$ on $\partial\Omega_2$, the boundary-value problem (P) is:

$$(P) \begin{cases} div\tilde{\sigma} + \underline{f}^g = \rho \underline{\ddot{u}} \text{ in } \Omega \\ \tilde{\sigma} = \frac{\partial w}{\partial \tilde{\epsilon}}(\tilde{\epsilon}, \tilde{\alpha}) \text{ in } \Omega \\ \tilde{\epsilon}(\underline{u}) = \frac{1}{2}(grad\underline{u} + (grad\underline{u})^T) \text{ in } \Omega \\ \underline{u} = \underline{u}^g \text{ on } \partial\Omega_2 \\ \tilde{\sigma}.\underline{m} = \underline{F}^g \text{ on } \partial\Omega_1 \end{cases} \tag{5}$$

where $\underline{m}$ is the outward unit normal to $\partial\Omega$. Then the quasi-static problem (P_{aux}) associated to (P) is found simply by taking a null right-hand side in the first equation of (5). Following this, $(\dot{P}_{aux})$ and $(\ddot{P}_{aux})$, respectively the auxiliary rate problem and the auxiliary acceleration problem, are written simply by derivating (P_{aux}) once and twice with respect to time. In that case, $\ddot{\tilde{\sigma}}$ is written as:

$$\ddot{\tilde{\sigma}} = \tilde{\tilde{E}}^{w} : \ddot{\tilde{\epsilon}} + \tilde{V}(\tilde{\epsilon}, \tilde{\alpha}) : \dot{\tilde{\epsilon}} + \tilde{C}(\tilde{\epsilon}, \tilde{\alpha}) \tag{6}$$

where $\tilde{V}$ and $\tilde{C}$ are non rate terms functions.

BIFURCATION OF THE ACCELERATION BOUNDARY-VALUE PROBLEM

The solid is supposed to be in quasi-static equilibrium i.e. (P_{aux}) is considered, and till the onset of localisation the mechanical state is homogeneous. It means that an approximation of the solution $\underline{u}$ is looked for; the solution is said to be quasi-static. By taking null body forces, the equilibrium equations in $(\ddot{P}_{aux})$ are written as:

$$div(\tilde{\tilde{E}}^{w} : \ddot{\epsilon} + \tilde{V} : \dot{\epsilon} + \tilde{C}) = 0 \text{ in } \Omega \tag{7}$$

A linear perturbation stability analysis

The stability analysis is performed involving a perturbation of the solution $\dot{\underline{u}}^0$ of $(\dot{P}_{aux})$. The linear perturbation $\dot{\underline{u}}^1$ is a planar wave of the form:

$$\begin{cases} \dot{\underline{u}} = \dot{\underline{u}}^0 + \dot{\underline{u}}^1 \\ \dot{\underline{u}}^1(\underline{x}, t) = \underline{F}exp(ik\underline{x}.\underline{n} + \omega t) \end{cases} \tag{8}$$

$|\underline{F}| \ll 1$, small initial perturbation ($|\underline{A}|$ denotes the norm of vector $\underline{A}$);
$\underline{n}$, propagation direction (unit); $\underline{k} = \mathrm{k}\underline{n}$, wave vector;
ω, characteristic pulse of the perturbation;
i is the complex number so that $i^2 = -1$.

As the material is perturbed by a rate perturbation, 'static' variables (i.e. non rate ones) do not have time to evolve and stay quasi-homogeneous. Injecting (8) into (7) one obtains:

$$\{\underline{n}.[\tilde{\tilde{E}}^{w} + \frac{1}{\omega}\tilde{V}].\underline{n}\}.\dot{\underline{u}}^1(t = 0) = 0 \tag{9}$$

This is the condition for which (7), i.e. locally $(\ddot{P}_{aux})$, has two solutions ($\dot{\underline{u}}^0$ and $\dot{\underline{u}}^1$), i.e. bifurcates. A non trivial solution of (9) is obtained for:

$$\exists?(\underline{n}, \tilde{\epsilon}^0, \tilde{\alpha}^0) \quad \exists?\omega > 0 \quad det(\underline{n}.\{\tilde{\tilde{E}}^{w} + \omega^{-1}\tilde{V}\}.\underline{n}) = 0 \tag{10}$$

One can notice that if bifurcation conditions for $(\dot{P}_{aux})$ are investigated with that analysis, the obtained criterion is $\det(\underline{n}.\tilde{\tilde{E}}^{w}.\underline{n})=0$, which is excluded because of the definite positiveness of $\tilde{\tilde{E}}^{w}$.

A QUANTITATIVE DEFINITION OF (QUASI)-STATICITY

A solution of (P) is said to be (quasi)-static when the r.h.s. of the linear momentum balance (i.e. the inertia terms), can be neglected. This can be associated with a quantitative definition given below:

$$\begin{cases} \underline{u} \text{ is a (quasi)-static solution of } (P) \text{ when} \\ \rho|\ddot{\underline{u}}| < \gamma \text{ where } \gamma \text{ is a critical norm to be defined} \\ \rho \text{ denotes the mass density.} \end{cases} \tag{11}$$

For a given displacement rate, γ can be linked to a critical kinetic power as proposed by Cherukuri and Shawki [3]. If the condition (11) is fulfilled, (P) is no longer to be solved; instead (P_{aux}) is to be considered.

LOCALISATION OF DEFORMATION AS A LOSS OF (QUASI)-STATICITY

Linking the two latter questions (bifurcation of $(\ddot{P}_{aux})$ and quantitative definition of (quasi)-staticity) the localisation of deformation is postulated to be a physical phenomenon showing a local transition from a (quasi)-static equilibrium to a dynamic one. The whole structure stays however in static equilibrium under quasi-static loadings.

$(\ddot{P}_{aux})$ bifurcates if the following condition is satisfied:

$$\exists ?\omega > 0, \ \exists ?(\tilde{\epsilon}^0, \tilde{\alpha}^0) \qquad det(\underline{n}.\{\underset{\approx}{E}^{w} + \omega^{-1}\tilde{V}\}.\underline{n}) = 0 \tag{12}$$

The loss of (quasi)-staticity (or a sufficient condition for that) is, at the perturbation incipience:

$$\left\{ \begin{array}{l} \rho|\underline{\ddot{u}}^1| \geq \gamma \\ |\underline{\ddot{u}}^1| = \omega|\underline{F}| \end{array} \right. \tag{13}$$

So $\omega = \frac{\gamma}{\rho|\underline{F}|}$, and the localisation condition is:

$$\exists ?(\underline{n}, \tilde{\epsilon}^0, \tilde{\alpha}^0, |\underline{F}|) \ \ det(\underline{n}.[\underset{\approx}{E}^{w}(\tilde{\epsilon}^0, \tilde{\alpha}^0) + \frac{\rho|\underline{F}|}{\gamma}\tilde{V}(\tilde{\epsilon}^0, \tilde{\alpha}^0)].\underline{n}) = 0 \tag{14}$$

It is immediate to notice that this criterion depends on ρ, $|\underline{F}|$ and γ.

- the ρ-dependence and the $|F|$-dependence are linked and suggest that the material will not localise if a too small perturbation is introduced compared to its density.
- the γ-dependence reflects a material feature (via ρ) and a loading rate one (via $\underline{\dot{u}}$) as γ can be linked to a critical kinetic energy $e_c = \frac{1}{2}\rho|\underline{\dot{u}}|^2$. The γ-dependence would then exclude a material classification based only on their behaviour at the onset of the localisation: only a classification based on the material and the loading rate would be relevant. This remark follows the conclusions of Bataille and Kestin [8].

CONCLUSION

A new approach of modelling the physical phenomenon of localisation of deformation is presented. It is viewed as a local transition from a (quasi)-static behaviour to a dynamic one. The boundary-value problem is to be treated as dynamic after the onset of localisation.

Apart from this new definition, a localisation criterion is now given for rate dependent materials, which was either excluded by applying Rice's approach [1] or restricted to particular viscid solids (dynamic context studied by Loret and Prevost

[4], reversible part of the strain neglected by Anand et al. [5]).

Considering the dynamic boundary-value problem for the rate-dependent material linked to the post-localisation behaviour, some difficulties stemming from the rate-independent models can be lifted. These are a null width of the localisation band and a loss of objectivity of the Finite Elements response. It is known that for rate-dependent materials in a dynamic context [4], a band width naturally appears. Also the mesh sensitivity is suppressed [4] whereas in the quasi-static context it may be only reduced as observed by Forest and Cailletaud [2].

SYMBOLS USED

a denotes a scalar
$\underline{a}$ denotes a vector
$\tilde{a}$ denotes a second order tensor
$\tilde{\tilde{a}}$ denotes a fourth order tensor

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