DEVELOPMENT OF BATDORF'S APPROACH FOR MULTIAXIAL FRACTURE OF CERAMICS

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Two basic equations of brittle fracture statistics are derived theoretically. Based on them conclusions are reached on the non-equivalence of Batdorf's flaw density and orientation approach and Evans' multiaxial elemental strength method, which are commonly employed to evaluate the reliability of ceramic components in multiaxial states of stress. Previous work oversimplified the integral limits of the orientation integration in Batdorf's approach and resulted in the conclusion of equivalence of Batdorf's and Evans' methods. The details in Batdorf's and Evans' derivations are also discussed.

INTRODUCTION

Evans' multiaxial elemental strength method (1) and Batdorf's flaw density and orientation approach (2) are commonly employed to evaluate the reliability of ceramic components in multiaxial stress states and believed to be equivalent (3). In the present work the non-equivalence of these two approaches is proved.

THE PROBABILITY OF HAVING AT LEAST ONE MICROCRACK WITH STRENGTH SMALLER THAN σ_e IN A SOLID

Statistical Description of Microcracks in a Solid of Volume *V*

Microcracks in a solid distribute randomly with respect to spatial location, orientation, and strength, which are mutually independent. By denoting $g_0(S)$ as the probability density function of strength (S) distribution, Ω as the solid angle characterizing the orientation of a orientation, respectively, the probability that a microcrack with both a random orientation and an arbitrary spatial location as well as a strength equal to/smaller than the effective

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stress σ_e (as a function of principal stresses σ_1 ,σ_2 , σ_3, and crack orientation Ω) applied on it exists in the solid is given as:

$$p_1 = \frac{1}{V}\int_V \frac{1}{4\pi} \int_{\Omega_0=4\pi} \int_0^{\sigma_e} g_0(S) \cdot dS \cdot d\Omega \cdot dV \tag{1}$$

Statement 1. The Probability of Having Exactly *n* Arbitrary Microcracks in a Solid

Let it be assumed that: (1). an independent number of flaws gives strengths $S \in [0, \infty)$ in non-overlapping volumes; (2). the probability that such a microcrack occurs in a differential volume element dV is proportional to dV when dV is small; and (3). the probability that dV contains two or more microcracks is much smaller than the probability that there is only one microcrack in dV when dV is small enough. Then the probability of having n microcracks with arbitrary strength $S \in [0, \infty)$ in a solid of volume V obeys Poisson distribution as follows by V_0 as the mean volume occupied by each microcrack:

$$p_{2,n}(V) = \frac{(V/V_0)^n \cdot \exp(-V/V_0)}{n!} \tag{2}$$

Statement 2. The Probability of Having *k* Microcracks with Strength Smaller than the Effective Stress σ_e $(0 \leq \sigma_e < \infty)$ in *n* Existing Microcracks of Arbitrary Strength $S \in [0, \infty)$

The probability of having k microcracks with strengthes equal to or smaller than the effective stress σ_e $(0 \leq \sigma_e < \infty)$ in n existing microcracks of arbitrary strength $S \in [0, \infty)$ is given as

$$p_3 = \begin{cases} 0 & (k > n) \\ \dfrac{n!}{k! \cdot (n-k)!} \cdot p_1^k \cdot (1-p_1)^{n-k} & (0 \leq k \leq n) \end{cases} \tag{3a,b}$$

Proof of Statement 1 will be reported elsewhere in detail to limit the length of this presentation; while Statement 2 is self-evident according to the binomial distribution for Bernoulli trials.

Statement 3. The Probability of Having Exactly k Microcracks with Strengthes Smaller than σ_e in a Solid

Under the Poisson's postulates as stated in Statement 1, the probability of having k microcracks with strengthes equal to or less than σ_e $(0 \leq \sigma_e \leq \infty)$ in a solid of volume V obeys the following Poisson distribution:

$$p_{4,k}(V) = \frac{(p_1 \cdot V/V_0)^k \cdot \exp(-p_1 \cdot V/V_0)}{k!} \tag{4}$$

Proof of Statement 3. According to the concept of conditional probability,

$$p_{4,k}(V) = \sum_{n=0}^{\infty} p_{2,n}(V) \cdot p_3 = \sum_{n=k}^{\infty} C_n^k \cdot p_1^k \cdot (1-p_1)^{n-k} \cdot \left[\frac{(V/V_0)^n \cdot \exp(-V/V_0)}{n!} \right]$$
$$= \frac{(p_1 \cdot V/V_0)^k \cdot \exp(-p_1 \cdot V/V_0)}{k!} \tag{5}$$

Statement 4. The Probability of Having at Least One Microcrack with Strength Smaller than σ_e in a Solid

Under the Poisson's postulates as stated in Statement 1 and by assuming a ***uniform distribution*** of microcracks with respect to spatial location and orientation, the probability of having ***at least*** one microcrack with strength(S) equal to or less than σ_e ($0 \le \sigma_e \le \infty$) in a solid of volume V is given as:

$$p_{5,1}(V) = 1 - \exp\left[-\frac{1}{V_0} \int_V \frac{1}{4\pi} \int_{\Omega_0 = 4\pi} \int_0^{\sigma_e} g_0(S) \cdot dS \cdot d\Omega \cdot dV \right] \tag{6}$$

Proof of Statement 4. According *to* Statement 3, the probability of having ***at least*** one microcrack with strength equal to or less than σ_e ($0 \le \sigma_e \le \infty$) in a solid of volume V is given as:

$$p_{5,1}(V) = \sum_{k=1}^{\infty} p_{4,k}(V) = 1 - p_{4,0}(V) = 1 - \exp(-p_1 \cdot V/V_0) \tag{7}$$

Inputting Eq.(1) into Eq.(7) reduces to Eq.(6). Furthermore, by assuming that $g_0(S) = m \cdot \sigma_0^{-m} \cdot S^{m-1}$ ($m, \sigma_0 > 0$ are material constants.), Eq.(6) is reduced to the well known multiaxial Weibull theory in reference (4):

$$p_{5,1}(V) = 1 - \exp\left[-\frac{1}{V_0} \int_V \frac{1}{4\pi} \int_{\Omega_0 = 4\pi} \left(\frac{\sigma_e}{\sigma_0} \right)^m \cdot d\Omega \cdot dV \right] \tag{8}$$

It is worth emphasizing that the derivations of Eq.(6) and Eq.(8) are only based on pure probability theory rather than on the weakest-link postulate. In other word, the multiaxial Weibull theory (Eq.(8)) is in its nature not a weakest-link model.

THE CUMULATIVE FAILURE PROBABILITY OF A SOLID UNDER WEAK-EST-LINK POSTULATE

Statement 5. The Cumulative Failure Probability of a Solid under Weakest-Link Postulate

Under the Poisson's postulates as stated in Statement 1 and the assumption of a ***uniform distribution*** of spatial locations of microcracks in a solid, the cumulative probability of failure of a solid of volume V is given as

$$P = 1 - \exp\left[-\frac{1}{V_0} \int_V dV \int_0^\infty F(\sigma_e, S) \cdot g_0(S) \cdot dS \right] \tag{9}$$

where $F(\sigma_e, S)$ denotes the probability of an existing microflaw with a strength between S and $S+dS$ in the differential volume dV initiates fracture under an effective stress σ_e.

<u>Proof of Statement 5.</u> Under the assumption of a uniform distribution of flaws with respect to spatial location and according to the concept of conditional probability, the probability, dq, that a microcrack having a strength between S and $S+dS$ not only exists in a differential volume dV but also initiates fracture under the effective stress σ_e is given by:

$$dq = \frac{1}{V} \cdot dV \cdot g_0(S) \cdot dS \cdot F(\sigma_e, S) \tag{10}$$

where dV/V defines the probability of existence of a microflaw with an arbitrary strength S following $0 \le S < \infty$ in a differential volume dV, $g_0(S) \cdot dS$ defines the probability that an existing microflaw has a strength between S and $S+dS$.

Then the probability, q, that a microflaw with a strength S following $0 \le S < \infty$, not only exists in the volume V but also initiates fracture, is as follows:

$$q = \int dq = \frac{1}{V} \cdot \int_V dV \int_0^\infty F(\sigma_e : S) \cdot g_0(S) \cdot dS \tag{11}$$

The weakest-link postulate suggests that the failure probability of a solid containing n flaws, F_n, is given by:

$$F_n = 1 - (1 - q)^n \tag{12}$$

Under the Poisson postulates as given in Statement 1, the actual number n of flaws in a solid follows the Poisson distribution as given by Eq.(2). Then the conditional probability that a solid of volume V contains n flaws and failures under external loading, P_n, is:

$$P_n = p_{2,n}(V) \cdot F_n \tag{13}$$

The cumulative probability of failure for a solid containing an arbitrary number of flaws in its total volume (V), P, is now found by summing the term P_n over all possible numbers of flaws($n=0, 1, 2, \ldots, \infty$):

$$P=\sum_{n=0}^{\infty}P_n=\sum_{n=0}^{\infty}p_{2,n}(V)\cdot F_n=1-\exp\left(-q\cdot V/V_0\right) \tag{14}$$

Inputting Eq.(11) into Eq.(14) results in Eq.(9), which is the fundamental equation of the weakest-link fracture statistics. Moreover, under the normal tensile stress criterion (4), the analytic solution to $F(\sigma_e, S)$ in Eq.(9) has been obtained by Lei and Dahl (5) as below:

$$F(\sigma_e,S)=\begin{cases}0 & (S>\sigma_1\geq\sigma_2\geq\sigma_3)\\ 1-\dfrac{2}{\pi}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\Phi}\cdot d\beta & (\sigma_1\geq S>\sigma_2\geq\sigma_3)\\ 1-\dfrac{2}{\pi}\cdot\displaystyle\int_{Arc\cos\sqrt{\frac{S-\sigma_3}{\sigma_2-\sigma_3}}}^{\frac{\pi}{2}}\sqrt{\Phi}\cdot d\beta & (\sigma_1\geq\sigma_2\geq S>\sigma_3)\\ 1 & (\sigma_1\geq\sigma_2\geq\sigma_3\geq S)\end{cases} \tag{15a,b,c,d}$$

$$\Phi=\frac{(S-\sigma_3)-(\sigma_2-\sigma_3)\cdot Cos^2\beta}{(\sigma_1-\sigma_3)-(\sigma_2-\sigma_3)\cdot Cos^2\beta} \tag{16}$$

DICUSSION

As has been just proved, Eq.(6) (and hence the multiaxial Weibull Theory as in Eq.(8)) is neither a weakest-link model nor an equation of the cumulative failure probability. Only when the effective stress σ_e is independent of crack orientation (e. g. the maximum principal stress criterion $\sigma_e=\sigma_1$), can Eq.(6) be reduced to:

$$p_{5,1}(V)=1-\exp\left[-\frac{1}{V_0}\int_V\int_0^{\sigma_e}g_0(S)\cdot dS\cdot dV\right] \tag{17}$$

However, Evans' approach in reference (1) is just based on Eq.(17) and a crack orientation dependent effective stress σ_e is still employed by introducing the relations as below:

$$\begin{cases}g_0(S)\cdot dS=\dfrac{1}{4\pi}\cdot\displaystyle\int_{\Omega_0=4\pi}g(\sigma_e)\cdot d\sigma_e\\ \sigma_e=\sigma_e(\sigma_1,\sigma_2,\sigma_3,\Omega)=\sigma_1\cdot\chi(\dfrac{\sigma_2}{\sigma_1},\dfrac{\sigma_3}{\sigma_1},\Omega)\\ d\sigma_e=\chi(\dfrac{\sigma_2}{\sigma_1},\dfrac{\sigma_3}{\sigma_1},\Omega)\cdot d\sigma_1\end{cases} \tag{18a,b,c}$$

Obviously, Eq.(18c) should be replaced by:

$$d\sigma_e = \sum_{i=1}^{3} \frac{\partial \sigma_e}{\partial \sigma_i} \cdot d\sigma_i + \frac{\partial \sigma_e}{\partial \Omega} \cdot d\Omega \tag{19}$$

In Batdorf's method in reference (2) it is assumed that:

$$F(\sigma_e, S) = \int_{\Omega_0 = 4\pi} H(\sigma_e, S) \cdot \frac{\Omega}{4\pi} \tag{20}$$

$$H(\sigma_e, S) = \begin{cases} 1 & (\sigma_e \geq S) \\ 0 & (\sigma_e < S) \end{cases} \tag{21a,b}$$

In fact, inputting Eq.(21a,b) into Eq.(20) results in

$$F(\sigma_e, S) = \begin{cases} 1 & (\sigma_e \geq S) \\ 0 & (\sigma_e < S) \end{cases} \tag{22a,b}$$

which is not true for the case of normal tensile stress criterion, i. e. Eq.(15a,b,c,d).

Finally, we have already found essential differences between Eq.(6) (or Eq.(8)) and Eq.(9) in the ***tension and compression mixed triaxial stress states*** while good equivalence in ***triaxial tension*** for some fracture criteria, which will be reported in detail elsewhere.

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