# Fatigue Crack Growth Effect on the Dynamic Behaviour of Frame Structures

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***ABSTRACT.*** *In this paper the stiffness matrix for a straight two-node cracked Timoshenko beam element is derived. The equation of motion of the complete system includes translational and rotational mass matrices. The vibration characteristics of plane frame structures with a single edge crack are investigated using a modified line-spring model. The natural frequencies and the corresponding mode shapes are determined for edge cracks of different depths. By using an extension of the Paris crack propagation law, the fatigue evolution of cracked frame-structures and the determination of the bending moment redistribution is analysed and graphically illustrated. The retarding effect on the crack growth rate in the case of redundant structures subjected to repeated loading is pointed out.*

## INTRODUCTION

As is well known, a structure is designed to perform certain functions. It is put out of use when, after reaching a certain limit state, it is no longer meets the requirements for which it was devised. In the case of a cracked structural member some modes of failure (limit states) can be considered, such as compression instability buckling, ultimate plastic collapse, brittle fracture, fatigue, etc. The above different failure mechanisms may also affect one another. For a cracked structural element a main problem is to determine whether the dominant crack reaches the critical conditions in the interval between two following inspections under ordinary conditions of use. In the first case the structural element must be replaced or repaired, while in the second case the reliability of the structural element is decided during the subsequent inspection.

In order to predict the component life in such circumstances, of interest is the estimation of fatigue life based on the number of stress cycles at the stage of crack growth. Empirical formulas estimating the rate of growth of fatigue cracks have long been known for special cases. However, only the inclusion of stress intensity factor among the parameters affecting crack propagation makes possible a quantitative and qualitative analysis of the laws of crack growth under repeated loading.

To treat with the rate of crack growth as depending on the stress intensity factor, numerous relations have been proposed. All these relations can be considered as an extension of the Paris' formula. It is worth noting that, a crack on a structural member introduces a local flexibility which is a function of the depth crack. This flexibility changes the static and dynamic behaviour of the structures. The fracture mechanics approach can yield the local compliances due to the cracked sections for which more and more expressions of Stress Intensity Factors (SIFs) should be developed. The local flexibility of the cracked region of the structural element was put into relation with the SIFs. A general method for extending fracture mechanics through the compliance concept to the analysis of a structure containing cracked members was considered by Okamura et al. [1].

In this paper, the stiffness matrix for a straight two-node cracked Timoshenko beam element is derived. The equation of motion of the complete system in a fixed co-ordinate system includes translational and rotational mass matrices. The problem of determining the natural vibration frequencies and the associated mode shapes of a system always leads to solving an eigenvalue problem, where the mass and stiffness matrices are nearly symmetric and positive definite. A parametric study of a transverse open crack is carried out for various crack depths and the changes in eigenfrequencies as a function of crack position and fatigue crack growth is determined. By supposing that the external load takes all the values between zero and a fixed maximum value, the stress intensity factor range is calculated. During each cycle of loading a definite increment of crack length can be obtained. The crack length therewith increases and this new length must be taken as the initial length in the calculation for the next cycle.

## METHOD OF ANALYSIS

As a general rule, a crack in a beam element introduces a local flexibility that affects its static and dynamic behaviour. One of the objective of the paper is to determine the vibration characteristics of a uniform Timoshenko beam element with a single edge crack using a modified line-spring model. The governing matrix equation for free vibrations of the cracked beam is derived by assembly of the conventional cubic beam elements in conjunction with the modified line-spring model. The “springs” have the features of having two nodes and zero length. The resulting eigenvalue problems are solved to find the natural frequencies and the corresponding mode shapes of structures.

A pre-cracked bending specimen is modelled by one-dimensional beam elements and a line-spring representing the stiffness or compliance of a cracked part. Then the following finite element equation for a transient dynamic analysis is obtained:

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{C}\dot{\mathbf{U}}+\left(\mathbf{K}+\mathbf{K}_s\right)\mathbf{U}=\mathbf{F} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass matrix, the damping matrix and the stiffness matrix of the system, respectively, which are obtained using the usual finite element procedure. $\mathbf{U}$ and $\mathbf{F}$ are the vectors whose components are the nodal displacements and forces, respectively; $\dot{\mathbf{U}}$ and $\ddot{\mathbf{U}}$ are the vectors whose components are the nodal velocities and accelerations, respectively. $\mathbf{K}_s$ is the stiffness matrix of a line-spring in the extended form. Let us consider a model for evaluating the local stiffness matrix $\mathbf{k}_s$, as shown in Fig.1, which indicates dimensions and sign conventions for forces and corresponding displacements. Then the following relation is obtained:

$$\mathbf{F}_s = \mathbf{k}_s \mathbf{u}_s \tag{2}$$

where $\mathbf{F}_s$=[ $P_1$, $V_1$, $M_1$, $P_2$, $V_2$, $M_2$]$^T$ and $\mathbf{u}_s$=[ $u_1$ $w_1$ $è_1$ $u_2$ $w_2$ $è_2$]$^T$. The stiffness matrix of a line-spring $\mathbf{k}_s$ is given by Tharp [2] as follows:

$$\mathbf{k}_s = \begin{bmatrix} l_{mm}/D & 0 & -l_{mp}/D & -l_{mm}/D & 0 & l_{mp}/D \\ 0 & 1/l_{vv} & 0 & 0 & -1/l_{vv} & 0 \\ -l_{mp}/D & 0 & l_{pp}/D & l_{mp}/D & 0 & -l_{pp}/D \\ -l_{mm}/D & 0 & l_{mp}/D & l_{mm}/D & 0 & -l_{mp}/D \\ 0 & -1/l_{vv} & 0 & 0 & 1/l_{vv} & 0 \\ l_{mp}/D & 0 & -l_{pp}/D & -l_{mp}/D & 0 & l_{pp}/D \end{bmatrix} \tag{3}$$

where $D=ë_{pp}ë_{mm}–ë^2_{mp}$. In Eq. 3, $ë_{pp}$ , $ë_{mm}$ , $ë_{vv}$ are compliance expressions for extension, bending and shear, respectively and $ë_{mv}=0$, $ë_{pv}=0$ are compliances for the coupling of bending and shearing, extension and shearing respectively. The compliance matrix for this cracked member may be derived according to the theory presented by Okamura et al. [3] and Carpinteri et al. [4] as

$$l_{pp} = \frac{2\left(1-n^2\right)}{E}\int_0^A\left(\frac{K_{IP}}{P}\right)^2 dA,\quad l_{mm} = \frac{2\left(1-n^2\right)}{E}\int_0^A\left(\frac{K_{IM}}{M}\right)^2 dA$$
$$l_{vv} = \frac{2\left(1-n^2\right)}{E}\int_0^A\left(\frac{K_{IIV}}{V}\right)^2 dA,\quad l_{mp} = \frac{2\left(1-n^2\right)}{E}\int_0^A\left(\frac{K_{IP}}{P}\right)\left(\frac{K_{IM}}{M}\right) dA \tag{4}$$

In Eqs 4, $K_{IP}$ and $K_{IM}$ are mode I stress intensity factors caused by axial load P and bending moment M respectively, and $K_{IIV}$ is mode II stress intensity factor caused by shear force V. E and í are the Young’s modulus and the Poisson’s ratio, respectively. Cracked area is denoted by A and its infinitesimal increment dA is equal to Bda, where B is thickness and a is crack length. The following equations for $K_{IP}$, $K_{IM}$ and $K_{IIV}$ are utilized in the present paper:

$$K_{IP} = \frac{P}{BH}\sqrt{pa}F_p\left(x\right) \tag{5}$$

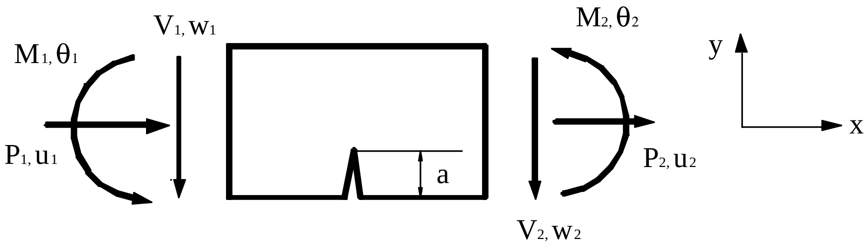


Figure 1. Model for evaluating the stiffness of a line spring.

$$K_{\mathrm{IM}} = \frac{6M}{BH^2}\sqrt{\pi a}F_m(\xi) \tag{6}$$

$$K_{\mathrm{IIV}} = \frac{V}{B\sqrt{H-a}}\sqrt{\pi a}F_v(\xi) \tag{7}$$

where $\xi = a/H$. The functions $F_p(\xi)$, $F_m(\xi)$, and $F_v(\xi)$, are given by Brown & Srawley [5] and Tharp [2].

## FATIGUE CRACK GROWTH CALCULATION

The concept of the damage tolerance design and increased demand for accurate component life predictions have provided growing demand for the study of fatigue crack growth in mechanical components. Cracks growing under opening or mode I mechanism is concerned with the traditional applications of fracture mechanics.

It should be noted that many service failures occur when cracks are subjected to mixed mode loadings. Various uniaxially loaded materials often contain randomly oriented defects and cracks which are subjected to a mixed mode state by virtue of their orientation with respect to the loading axis. Usually, mixed mode fatigue is characterized by crack propagation in a non-self similar manner. In other words, when subjected to mixed mode loadings, a crack changes its growth direction. Therefore, under mixed mode loading conditions, not only the fatigue crack growth rate is of importance, but also the crack growth direction. Several criteria can be found in the literature regarding the crack growth direction under mixed mode loading. Within the limits of linear elastic fracture mechanics, the driving force of crack propagation is known to be a function of the applied stress intensity factor range $\Delta K$. Among the developed relations for predicting the crack growth rate under cyclic loading, the well known Erdogan-Paris formula [6] is the simplest one. It is expressed as a function of an effective stress intensity factor as follows:

$$\frac{da}{dN} = C\left(\Delta K_{\mathrm{eff}}\right)^m \tag{8}$$

where $C$ and $m$ are empirical coefficients for the given material, Ä$K_{\text{eff}}$= $K_{max}$ –$K_{min}$ is the range of the combined mode I and II stress intensity factor for stress cycle and $N$ is the number of cycles. By supposing that the external load takes all the values between zero and a fixed maximum value and considering the combined mode I and II loadings, the Ä$K_{\text{eff}}$ =Ä$K_{\text{eq}}$ proposed by the authors of this paper is:

$$\mathbf{D}K_{\text{eff}} = \mathbf{D}K_{\text{eq}} = \sqrt{\left(\mathbf{D}K_{\text{IP}} + \mathbf{D}K_{\text{IM}}\right)^2 + 3\mathbf{D}K_{\text{IIV}}^2} \tag{9}$$

Under mixed mode conditions, it is assumed that deformations due to mode I and II loads are not interactive. The number of cycles required to propagate a crack from the initial size $a_i$ to some size $a_f$ may be obtained by integrating the relationship (8). For ferritic-perlitic steels the rate $da/dN$ of fatigue crack growth can be expressed by the following equation [7]:

$$\frac{da}{dN} = 6.9\times10^{-12}\left(\mathbf{D}K_{\text{eff}}\,MPa\sqrt{m}\right)^3 \tag{10}$$

The total number of cycles required for the crack extension from $î_0$ to $î_1$ is given by:

$$N = N(x_1) - N(x_0) = \int_{x_0}^{x_1} \frac{1}{6.9\times10^{-12}\left(\mathbf{D}K_{\text{eff}}\right)^3} \tag{12}$$

in which $î_0=a_0/H$ and $î_1=a_1/H$ are the initial and final dimensionless crack depths.

Consider a Timoshenko cracked frame structure having double fixed end with a crack at 0.1 m from the left clamped end as shown in Fig. 2. The height and the length of the frame are 1 m, whereas the thickness B and the depth H of the rectangular cross section are B = 0.05 m and H = 0.1 m, respectively. The elastic modulus E of the material, the Poisson's ratio and the maximum value of the external load are assumed to be E = 2.1 x $10^5$ MPa, í = 0.3, q = $10^5$ $\text{Nm}^{-1}$, respectively. The edge crack of initial length $î_0$= a/H = 0.001 has been supposed to exist before any loading application. Figure 2 shows the variation of the axial force, the shear force and the bending moment as a function of the dimensionless crack depth at the cracked section. It can be seen that the bending moment tends to zero when deeper cracks are considered. The trend in discussion is also illustrated in the following Fig. 4. Figure 3 shows the total number of load repetitions $N$ which must be applied for the crack growth from the initial length $î_0$= a/H = 0.001 to some size $î$. In particular, this figure shows the total number of load repetitions $N$ when the driving force of crack propagation is a function of the applied stress intensity factor range Ä$K_{\text{eff}}$ = Ä$K_{\text{IP}}$, depending on the axial force $P$, and when Ä$K_{\text{eff}}$ = Ä$K_{\text{IM}}$ depends on the bending moment $M$. Moreover, Fig. 3 illustrates the total number of load repetitions $N$ when Ä$K_{\text{eff}}$ = Ä$K_{\text{IP}}$ +Ä$K_{\text{IM}}$ as well as when the total number of load repetitions $N$ is a function of the Ä$K_{\text{eff}}$ =Ä$K_{\text{eq}}$ as proposed by the authors in Eq. (9). In Tab.1 it can be seen the numerical values of fatigue life calculation according to various criteria of loading combinations.

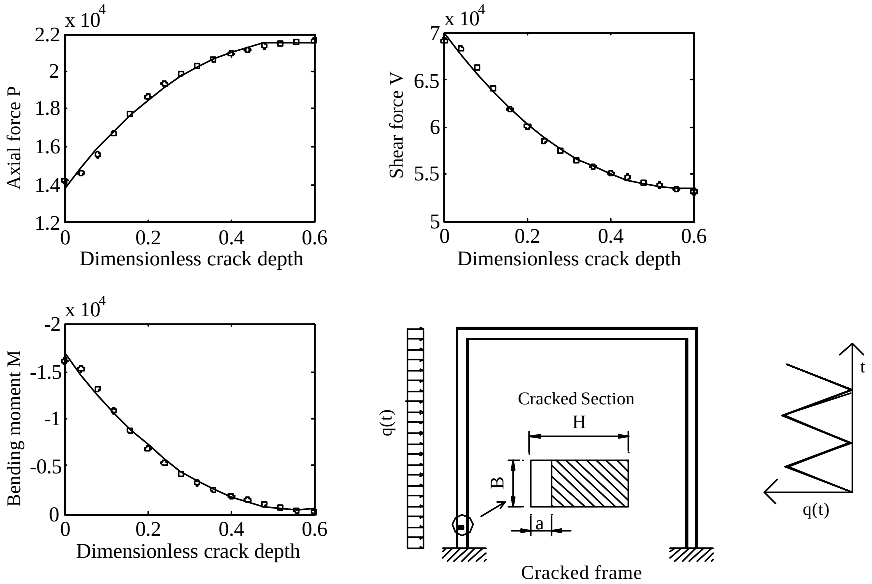


Figure 2. Variation of axial force, bending moment and shear force, at the cracked section, as a function of the dimensionless crack depth.

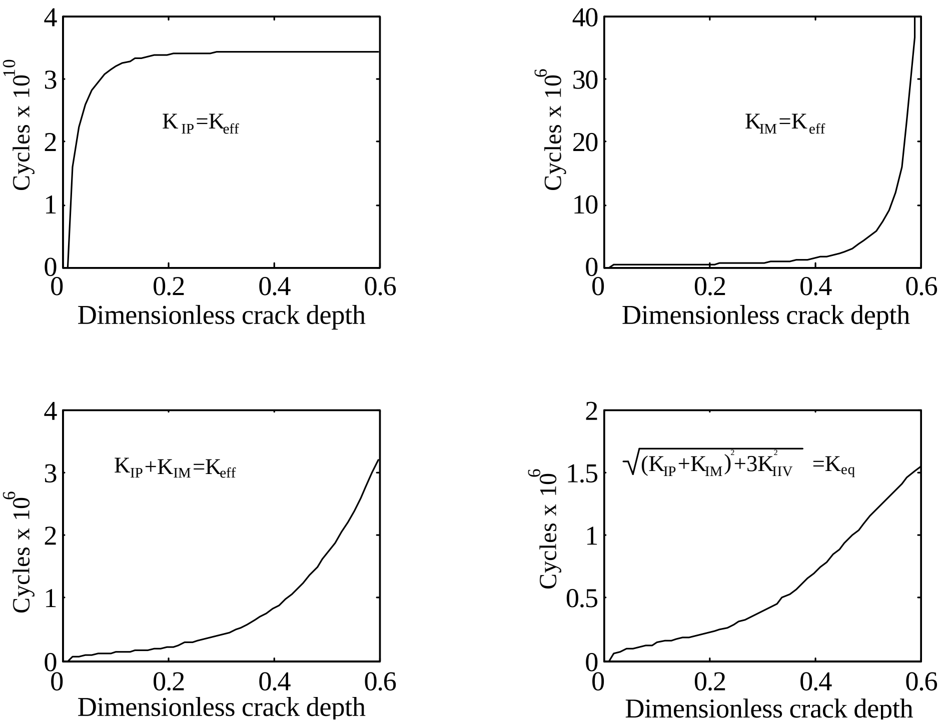


Figure 3. Fatigue cycles calculation according to various criteria of loading combinations.

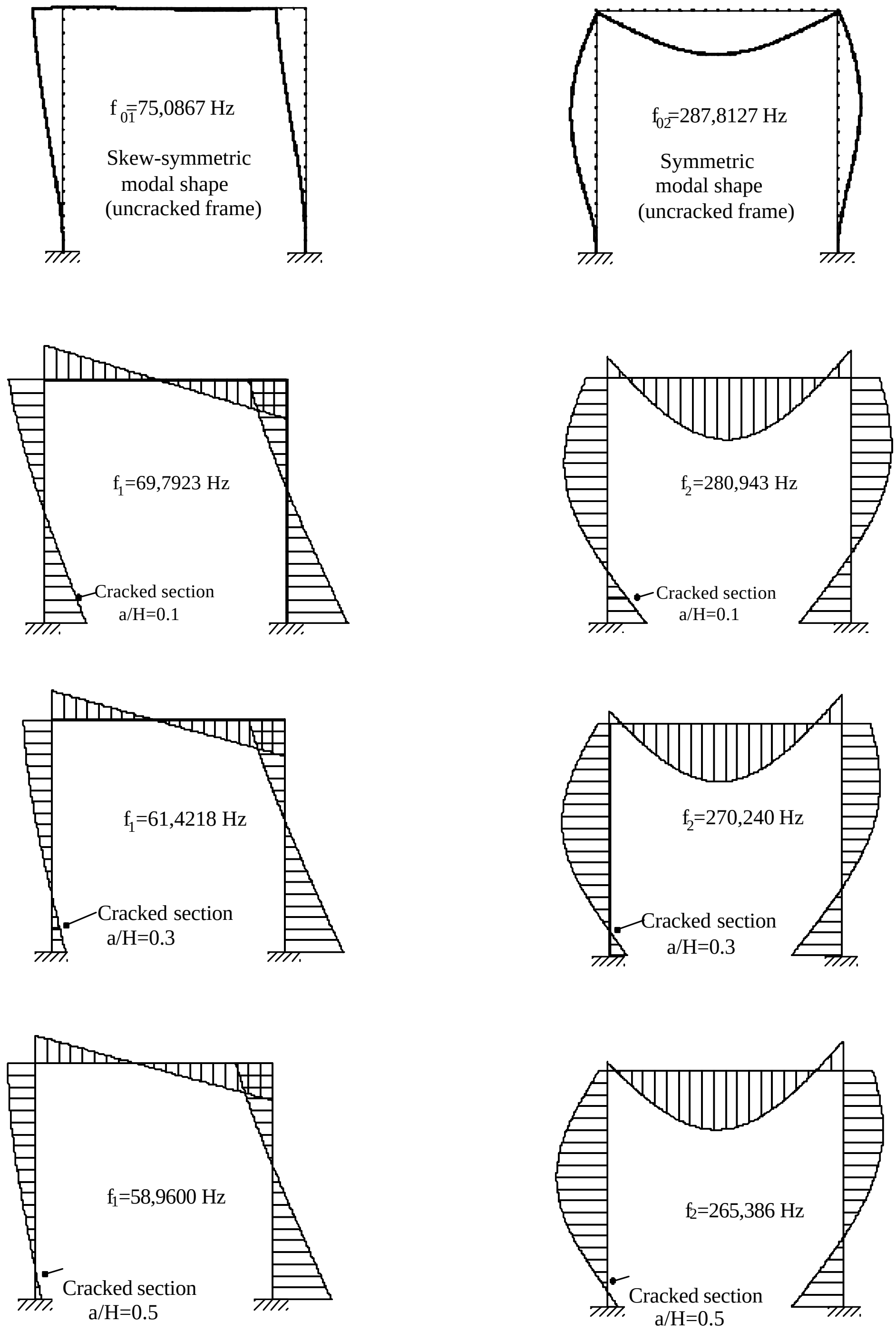


Figure 4. Bending moment redistribution for the first two modal shapes when various values of the crack depth are investigated.

Table 1. Numerical values of the cycles number N as a function of various criteria of loading combinations and for five dimensionless crack depths.

| **a/H** | **0.1044** | **0.2106** | **0.305** | **0.4112** | **0.5056** |
|---|---|---|---|---|---|
| **ÄK$_{eff}$ = ÄK$_{IP}$** (cycles x $10^{10}$) | **3.2069** | **3.3898** | **3.4199** | **3.4290** | **3.4312** |
| **ÄK$_{eff}$ = ÄK$_{IM}$** (cycles x $10^{6}$) | **0.1537** | **0.2637** | **0.5115** | **1.4149** | **4.7845** |
| **ÄK$_{eff}$ =ÄK$_{IP}$+ÄK$_{IM}$** (cycles x $10^{6}$) | **0.1459** | **0.2425** | **0.4273** | **0.9032** | **1.7520** |
| **ÄK$_{eff}$ = ÄK$_{eq}$** (cycles x $10^{6}$) | **0.1457** | **0.2400** | **0.4064** | **0.7472** | **1.1509** |

Figure 4 shows the first and second modal shapes of the uncracked frame as well as the corresponding bending moment distributions along the frame for three values of the dimensionless crack depth, *î* = a/H = 0.1, *î* = a/H = 0.3 and *î* = a/H = 0.5, obtained from a dynamic analysis. In Fig. 4, $f_{0i}$ (i=1,2) are the first two frequencies of the uncracked frame and $f_i$ the frequencies of cracked frames for three different crack depths. It should be noted that, when deeper cracks are considered, a migration of the bending moment from the centrally cracked zone towards the stiffest parts of the structure is produced.


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