Static Analysis and Fatigue of Cracked Circular Arches

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ABSTRACT*. This paper deals with static analysis and fatigue life prediction of cracked circular arches. The exact formulation of a curved beam finite element including axial extension and transverse shear effects is presented. The stiffness matrix is determined based on the exact solution of the elastic equilibrium problem. A simple two-node element with three degrees of freedom per node is obtained. The presence of a crack is modelled by a line-spring of appropriate stiffness. Using the Paris-Erdogan law, the fatigue cycles of cracked circular arches subjected to pulsating loads can be assessed.*

INTRODUCTION

The arch problem is of interest for a variety of engineering applications. The presence of a fracture at a certain cross section of a curved beam reduces the local stiffness. It should be remarked that, within the framework of one-dimensional beam theory, a fracture may give rise to discontinuities in axial and shear deformations as well as in slope. Thus, a compliance matrix has to be employed to relate generalised displacements to forces in the case of general loading. Recently, in order to model the structure for FEM analysis, a special finite element for a cracked Timoshenko beam has been developed by Viola et al. [1]. In the same work, a procedure for identifying cracks in structures by using modal test data has been also proposed.

In this paper, static analysis and fatigue of cracked circular arches are addressed. First, the exact formulation of a curved beam finite element for static analysis is presented, taking into account for bending moment, axial force and shear force effects. To this purpose, the exact solution of the elastic problem for a circular arch of uniform cross-section is derived. The approach proposed differs from those followed in [2] and [3]. By means of a suitable transformation, the homogeneous equilibrium equations in terms of displacements are put in an uncoupled form that can be easily solved. In particular, a third order differential equation involving only curvature is obtained. Based on the general solution of these uncoupled equations the exact shape functions for displacements as well as the exact stiffness matrix of the arch element are computed. Obviously, the resultant element is completely free of shear and membrane locking.

The fracture mechanics approach together with the concept of line-spring is used to model the presence of a crack. A local compliance matrix is introduced based on the well-known relationship between stress intensity factors and energy release rates. Then, a numerical procedure aimed at predicting fatigue life of cracked circular arches subjected to cyclic loading is established. The fatigue crack growth rate is expressed using the Paris-Erdogan law as a function of an effective value of the stress intensity factor range. Different forms of the effective stress intensity factor are considered and compared. Fatigue life prediction and internal forces redistribution caused by crack growing are illustrated by a numerical example.

STIFFNESS MATRIX FOR THE UNCRACKED FINITE ELEMENT

Consider a plane circular arch of uniform cross section and initial radius of curvature r (Fig. 1a). A curvilinear abscissa s spans the centroid axis and a local Cartesian co-ordinate system is introduced according to the tangent and the normal to the axis line. The arch is subjected to distributed external loads (per unit arch length), given as functions of the curvilinear abscissa in the local reference frame and denoted by p, q and m. The governing equations of the linear elastic arch problem are shown by Fig. 2 in the form of Tonti Diagram [4]. External and internal forces should satisfy the indefinite equilibrium conditions, where N, T and M are the axial force, the shear force and the bending moment. Then, invoking the principle of virtual work in the complementary form, the strain-displacement relations can be obtained, where u, v and φ are the tangential displacement, the radial displacement and the rotation, whereas ε, γ and χ are the axial, shear and bending strains. Finally, internal forces and strains are related by the constitutive equations. For a homogeneous isotropic elastic material these relations can be expressed as shown in Fig. 2, where E and G are Young's and shear moduli, A and I are the cross-section area and moment of inertia and $\Lambda = A / k_o$, being k_o the shear correction factor. Combining the above equations leads to a coupled system of second order differential equations (fundamental system of equations) to be solved in terms of displacements, together with suitable boundary conditions.

To solve the arch problem in the absence of distributed external forces, the fundamental equations are first put in an uncoupled form. With some manipulations, the following third order differential equation involving only curvature can be obtained

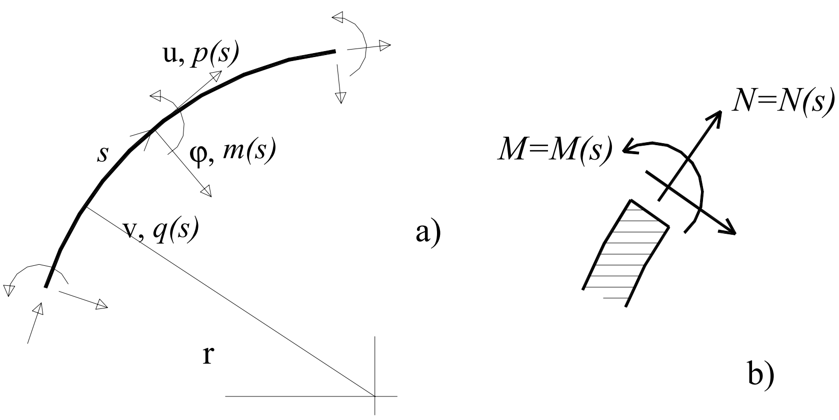

Figure 1. The plane circular arch.

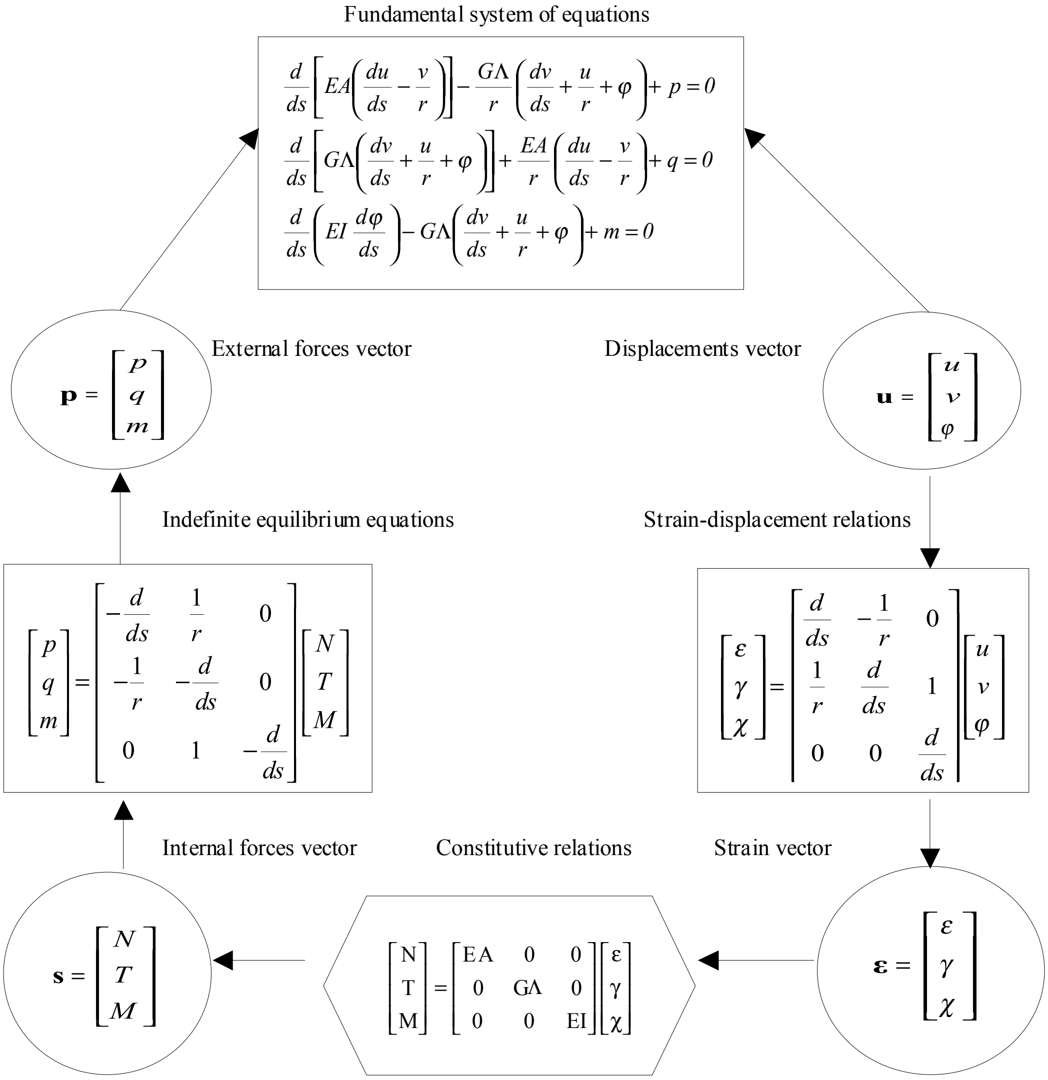

Figure 2. Arch problem in the form of Tonti Diagram.

$$\chi''' + \chi'/r^2 = 0 \tag{1}$$

together with two differential equations: the first involving tangential displacement and curvature, the second involving radial displacement and curvature. Then, the general solution of such system can be expressed as follows [5]:

$$\begin{aligned}
u &= \left(-C_2 + C_4 r\left(\frac{\lambda}{2}-\frac{I}{A}\right)\right)\cos\frac{s}{r} + \left(C_1 - C_3 r\left(\frac{\lambda}{2}-\frac{I}{A}\right)\right)\sin\frac{s}{r} + \frac{C_4\lambda}{2}s\,\sin\frac{s}{r} + \frac{C_3\lambda}{2}s\,\cos\frac{s}{r} - C_5 rs - C_6 r \\
v &= C_1\cos\frac{s}{r} + C_2\sin\frac{s}{r} - \frac{C_3\lambda}{2}s\,\sin\frac{s}{r} + \frac{C_4\lambda}{2}s\,\cos\frac{s}{r} - C_5 r^2 \\
\varphi &= C_3 r^2\sin\frac{s}{r} - C_4 r^2\cos\frac{s}{r} + C_5 s + C_6
\end{aligned} \tag{2}$$

where $\lambda = EI / G\Lambda + I / A + r^2$.

The six constants C_i in Eq. 2 can be determined according to the boundary conditions at the beam ends. For brevity the resultant expressions are not reported.

The strain energy stored in an arch element of length l is

$$\Phi = \frac{1}{2} EI \int_0^l \chi^2 ds + \frac{1}{2} G\Lambda \int_0^l \gamma^2 ds + \frac{1}{2} EA \int_0^l \varepsilon^2 ds \tag{3}$$

Based on the above results, strains can be expressed in terms of generalised nodal displacements $\mathbf{q}$ as

$$\varepsilon = \mathbf{N}_\varepsilon^T \mathbf{q}, \qquad \gamma = \mathbf{N}_\gamma^T \mathbf{q}, \qquad \chi = \mathbf{N}_\chi^T \mathbf{q} \tag{4}$$

where $\mathbf{N}_\varepsilon^T, \mathbf{N}_\gamma^T, \mathbf{N}_\chi^T$ collect the exact shape functions in the absence of distributed loads. Substituting Eq. 4 into Eq. 3 yields

$$\Phi = \frac{1}{2} \mathbf{q}^T \left(\mathbf{k}_\chi + \mathbf{k}_\gamma + \mathbf{k}_\varepsilon \right) \mathbf{q} = \frac{1}{2} \mathbf{q}^T \mathbf{k} \mathbf{q} \tag{5}$$

where

$$\mathbf{k} = \mathbf{k}_\chi + \mathbf{k}_\gamma + \mathbf{k}_\varepsilon = \int_0^l EI \mathbf{N}_\chi \mathbf{N}_\chi^T ds + \int_0^l G\Lambda \mathbf{N}_\gamma \mathbf{N}_\gamma^T ds + \int_0^l EA \mathbf{N}_\varepsilon \mathbf{N}_\varepsilon^T ds \tag{6}$$

is the element stiffness matrix.

FATIGUE LIFE PREDICTION

As it is well known, crack growth can take place under pulsating loads. From a practical point of view, the fatigue life prediction of structural members under cyclic loading is of great interest. The estimation of the number of stress cycles corresponding to the crack growth from an initial crack depth a_0 to a final crack depth a makes possible a quantitative analysis of the fracture process. In order to calculate the rate of crack growth under repeated loading, several models have been proposed in the course of the last forty years. All these models are closely related to the Paris-Erdogan law [6], which is based on the stress intensity factors knowledge. In fact, it has been experimentally shown that in metallic materials the stress intensity factor range ΔK mainly controls the fatigue growth of a crack, during the second stage of the fracture process, although many other factors may affect crack propagation.

According to the Paris-Erdogan law the fatigue growth rate da/dN can be expressed as a function of an effective stress intensity factor range ΔK_{eff} in the following form

$$\frac{da}{dN} = C\ \Delta K_{eff}^{n} \tag{7}$$

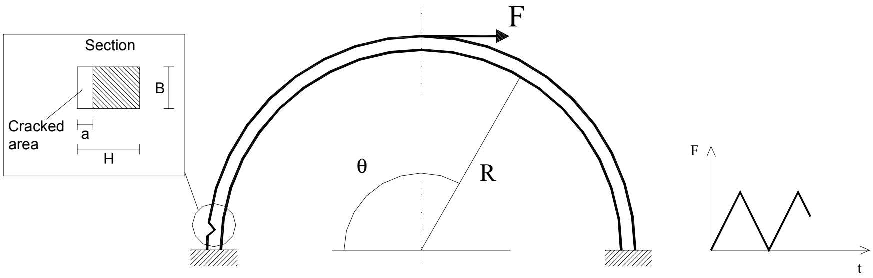

Figure 3. Clamped-clamped cracked circular arch under a pulsating force.

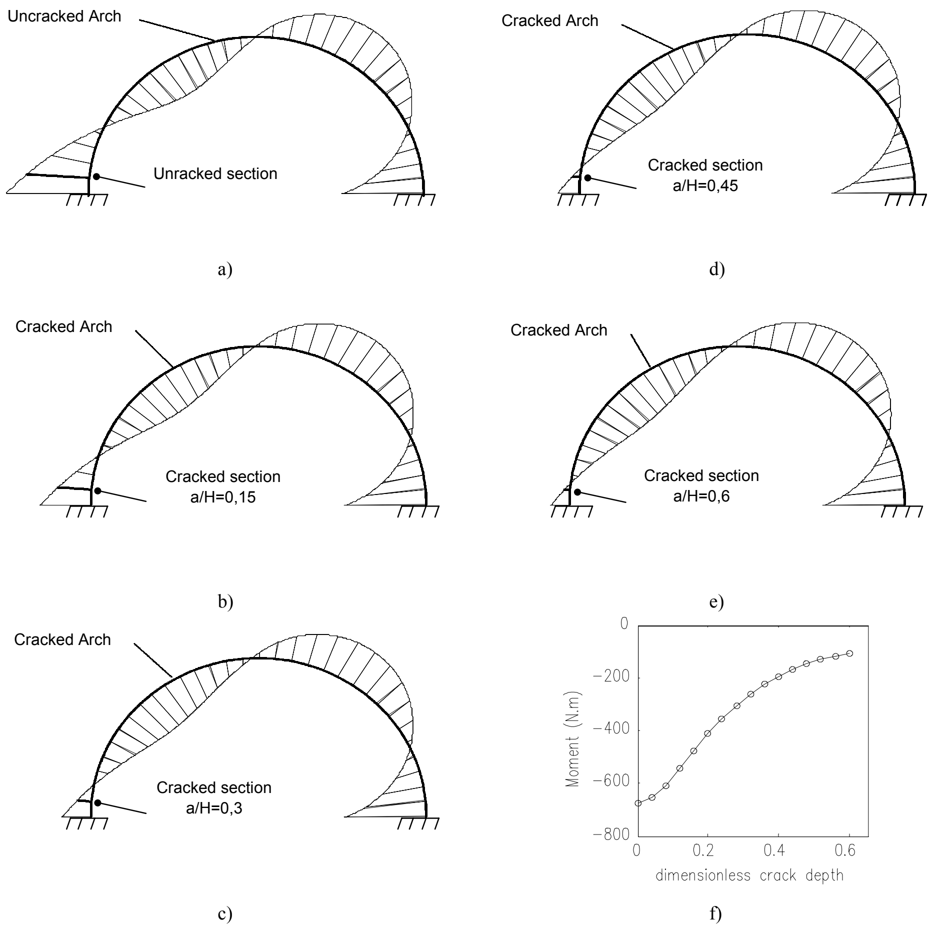

Figure 4. Bending moment redistribution due to crack growing.

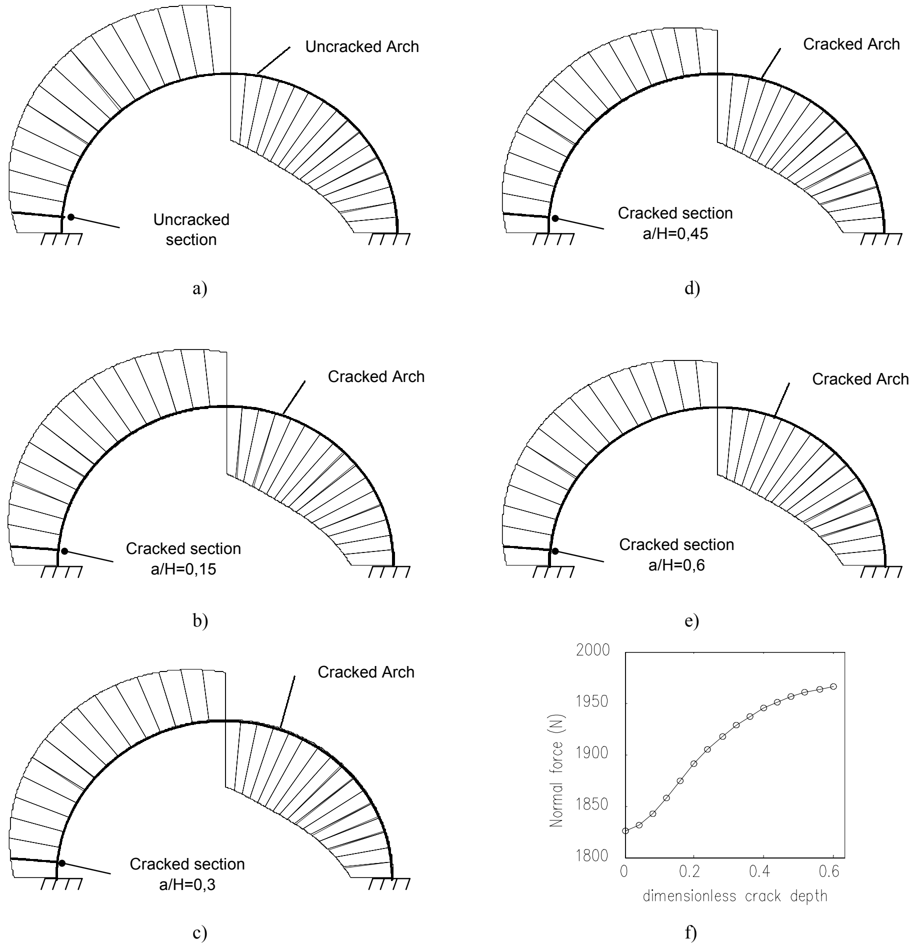

Figure 5. Axial force redistribution due to crack growing.

where coefficient C and exponent n account for the material effects and can be obtained from small specimens by a regulated procedure. It should be noted that no crack growth occurs when ΔK is less than its threshold value ΔK_{th}. Moreover, final fracture occurs when K_{max} reaches the fracture toughness.

The fatigue life can be estimated by integrating Eq. (7). In particular, it can be used to determine how many fatigue cycles $N(a)$ are required for the initial crack a_0 to reach a certain size

$$N(a) = \int_{a_0}^{a} \frac{da}{C(\Delta K_{eff})^n} \tag{8}$$

The above expression can be evaluated using a suitable numerical integration procedure. During each cycle a definite increment of the crack length can be computed. Then, the new crack length is taken as the initial length for the next cycle.

For illustration purposes a circular arch subjected to a single point pulsating force is considered, as shown by Fig. 3. The assumed data are: $R = 1$ m, B = 0.1 m, H = 0.1 m, $E = 2.1 \times 10^{11}$ N m^{-2}, $\nu = 0.3$. The amplitude of the pulsating force is taken as 5000 N and the crack is supposed to exist prior to the loading application. The material constants, C and n, in the Paris-Erdogan law are assumed to be 6.9×10^{-12} (with ΔK_{eff} expressed in MPa$\sqrt{m}$) and 3, respectively. The dimensionless initial depth of the crack is assumed to be $a_0/H = 0.01$. The finite element formulation developed in the previous section is used to model the arch and the presence of the crack is represented as suggested in [7]. Figures 4-6 show the internal forces redistribution caused by crack growing. In particular, Figures 4f, 5f and 6f show how the bending moment, the axial force and the shear force at the cracked section change as the crack depth increases. The number of fatigue cycles against the dimensionless crack depth is plotted in Fig. 7. Four different combinations of the stress intensity factors are used in evaluating the effective range ΔK_{eff}. The results obtained are compared in Fig. 7, where N is expressed in cycles $\times 10^6$.

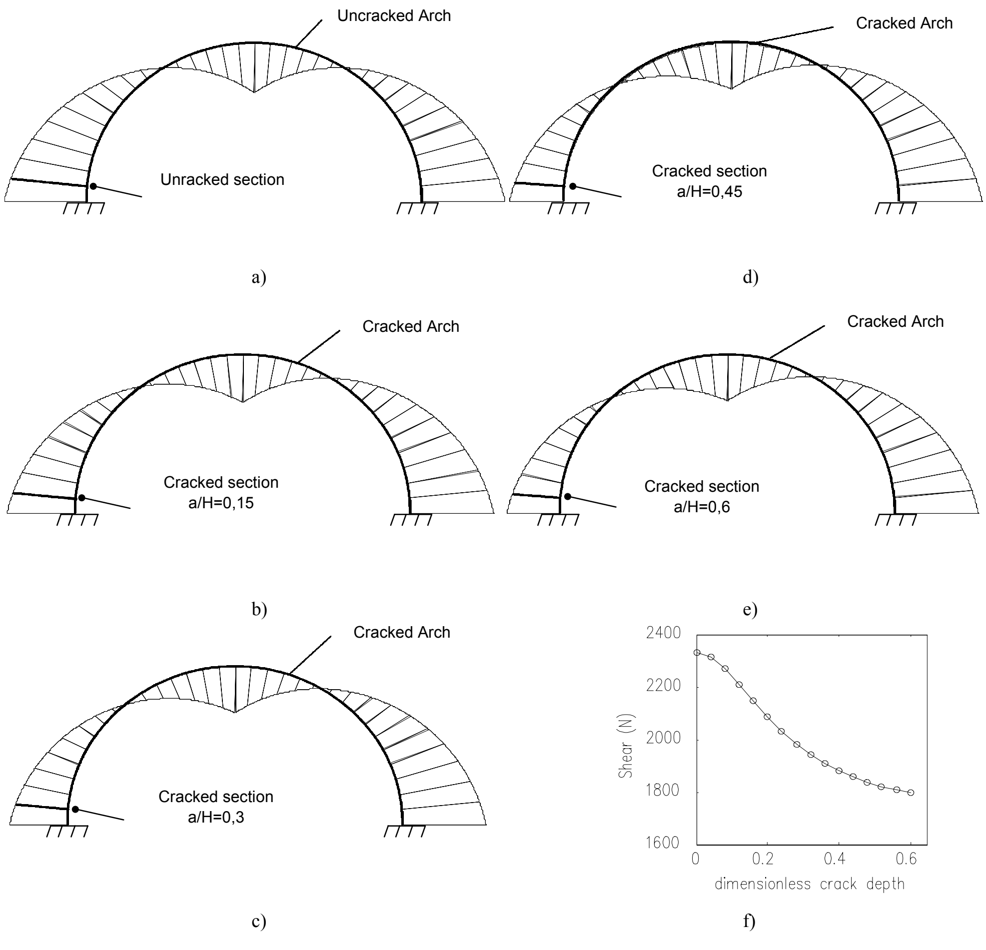

Figure 6. Crack effect on the shear force redistribution.

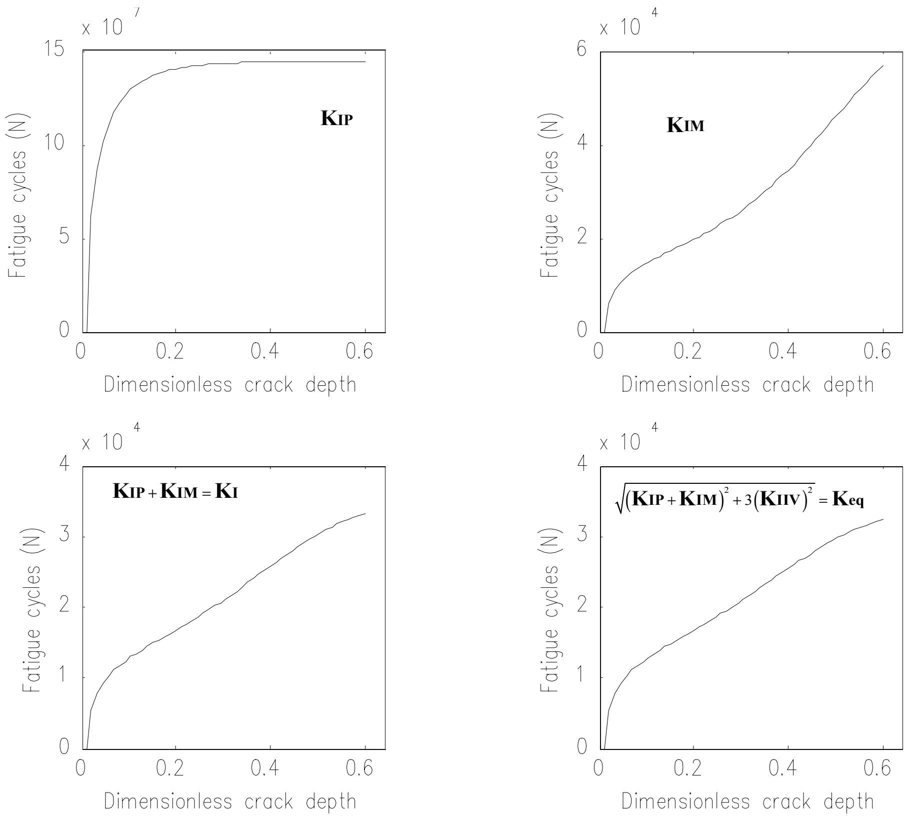

Figure 7. Fatigue cycles as a function of dimensionless crack depth for various stress intensity factor combinations.

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