# Why Threshold Stress Intensity Range is a Function of the Crack Length: an Explanation using Fractals

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***ABSTRACT.*** *It has long been recognized that the fatigue growth behaviour of cracks having a length comparable with the material microstructure size (the so-called short or small cracks) is remarkably different from that of long cracks. In particular, the threshold condition of fatigue crack growth is seen to be correlated to the crack length and the material microstructure. The well-known "Kitagawa diagram" describes the variation of the threshold stress intensity range against the crack length, showing the existence of a transition value of length beyond which the threshold of fatigue crack growth is governed by linear elastic fracture mechanics. In the present paper, the crack surface is firstly treated as a self-similar invasive fractal set (which is characterized by a uniform fractal dimension) and, owing to the fractional physical dimension of the fracture surface, the stress intensity factor is shown to be a function of the crack length. Consequently, the threshold stress intensity range is deduced to be a function of the crack length. Then the fractal dimensional increment is assumed to vary from 0 to 1 since, in the physical reality, the fractal dimension of the crack surface may change with the crack length. This allows us to put forward a new interpretation of the Kitagawa diagram within the framework of the fractal geometry.*

## INTRODUCTION

During last decades, the enhanced ability to detect and measure very short cracks and a great interest in using fracture mechanics methods for smaller and smaller crack sizes have pointed out the so-called "short (small) crack" problem (e.g. see Refs [1,2] for a review). Such cracks are characterized by an anomalous fatigue behaviour in comparison with that of their long counterparts, including: crack growth rate $\mathrm{d}a/\mathrm{d}N$ higher than what would be predicted by a long-crack curve, for a given stress intensity range $\Delta K$; often a decrease in $\mathrm{d}a/\mathrm{d}N$ with increasing $\Delta K$; crack growth at $\Delta K$ values lower than the long-crack threshold; crack growth rate strongly dependent on the material microstructure.

Standard threshold data of stress intensity range are commonly determined for long cracks. Hence, according to the implicitly governing similitude concept of Linear Elastic Fracture Mechanics (LEFM), such data are crack-size independent. Frost [3] firstly questioned the validity of LEFM-based threshold stress intensity range in the region of short cracks, showing that $\Delta K_{th}$ decreases with decreasing crack length. Kitagawa and Takahashi [4] later found that there exists a transition crack length below which $\Delta K_{th}$ is smaller than that for long cracks, and that such a length is dependent on the material microstructure. The dependence of the threshold stress intensity range on the crack length (crack-size effect) is commonly described by the $\Delta K_{th}$ against $a$ plot, which is known as the "Kitagawa diagram".

Some investigations have been carried out to interpret the Kitagawa diagram (e.g. see Refs [5-8]). In the present paper, the dependence of the threshold stress intensity range on the crack length is explained following a theoretical approach based on some fractal geometry concepts (e.g. see Refs [9,10]). Some applications of fractal geometry to size effect-related fatigue problems can be found in Refs [11-13]. A new definition of the stress intensity factor for *self-similar* fractal topologies (exploited to model crack surfaces) is used, and a general relationship of threshold stress intensity range $\Delta K_{th}$ versus crack length $a$ for *self-affine* fractal topologies is herein presented. Such a relationship, deduced according to multifractal concepts, offers a justification of the Kitagawa diagram. Some relevant experimental data [7] are analysed to show how to apply the theoretical approach proposed.

## KITAGAWA DIAGRAM ACCORDING TO THE ELHADDAD MODEL

As is mentioned above, the breakdown of LEFM-based threshold condition for short cracks is well summarised by the Kitagawa diagram (Fig. 1). According to the well-known ElHaddad model [6], the Kitagawa diagram is described by the following expression:

$$\Delta K_{th} = \frac{\Delta K_{th0}}{\sqrt{1+\frac{a_0}{a}}} \tag{1}$$

where $\Delta K_{th0}$ is the crack-size independent threshold stress intensity range for long cracks, $a_0$ is an intrinsic crack length defined as follows

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th0}}{\Delta \sigma_{th0}}\right)^2 \tag{2}$$

and $\Delta\sigma_{th0}$ is the fatigue limit for smooth specimens. The intrinsic crack length $a_0$ ranges from 1-10 μm for very high strength steels (yield stress $\sigma_y$ up to 2000 MPa) to 100-1000 μm for very low strength steels ($\sigma_y$ as low as 200 MPa).

Since the following relationship

$$\Delta\sigma_{th} = \frac{\Delta K_{th}}{Y\sqrt{\pi a}} \tag{3}$$

holds for the stress range at the threshold of crack growth ($Y$ is a dimensionless factor depending on the loading and the geometry of the cracked configuration), we can obtain the expression below by combining Eq.(1) and Eq.(3):

$$\Delta\sigma_{th} = \frac{\Delta K_{th0}}{Y\sqrt{\pi(a+a_0)}} \tag{4}$$

As is clearly shown in Eq. (4), the ElHaddad model introduces the concept of effective crack length which is the sum of the actual crack length and the intrinsic crack length.

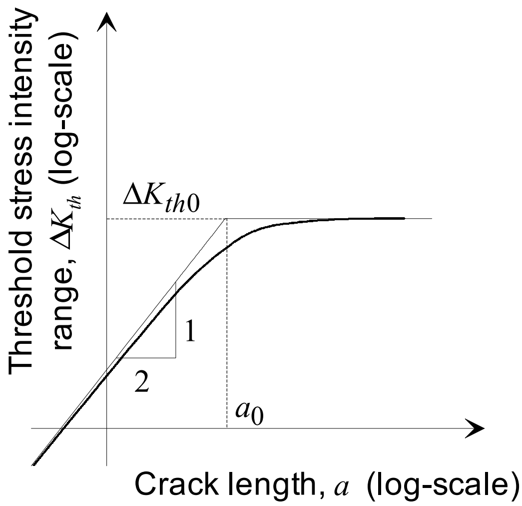



Figure 1. Schematic representation of the Kitagawa diagram: threshold stress intensity range $\Delta K_{th}$ as a function of the crack length $a$.

## THRESHOLD STRESS INTENSITY RANGE FOR FRACTAL CRACKS

Several theoretical investigations have been carried out in the field of fracture mechanics by considering the fractal nature of materials (e.g. see Ref. [14] for a review). In Ref. [15], for instance, fractal geometry (see Ref. [9] for basic concepts) has been exploited to explain size effect on fracture energy, by treating a fracture surface as an invasive fractal set, i.e. a set with a dimension higher than that of the Euclidean domain where it is contained. That applies to mathematical fractals (also called *self-similar fractals*) which are characterized by a *uniform* fractal (*monofractal*) dimension.

However, Mandelbrot [10] pointed out a *non-uniform* (*multifractal*) scaling of the natural fractals (also called *self-affine fractals*), different from the uniform one of the mathematical fractals. Accordingly, a transition from a fractal regime for small structures to a Euclidean one for structures large enough with respect to a characteristic material length has been considered, and a multifractal scaling law for fracture energy has been proposed [16].

***Self-Similar Fractal Cracks***

The modelling of a crack as a mathematical self-similar invasive fractal curve, such as the von Koch curve (e.g. see Ref. [9]), yields the definition of a renormalised (scale-invariant) threshold stress intensity range $\Delta K_{th}^{*}$ (details can be found in Refs [12,13]). Accordingly, the nominal stress intensity range $\Delta K_{th}$ turns out to be a function of the crack length through a power law:

$$\Delta K_{th} = \Delta K_{th}^{*}\, a^{\frac{d}{2}} \tag{5}$$

$d$ being the fractal increment with respect to the Euclidean domain where the fractal set is contained (*monofractal approach*). Note that $\Delta K_{th}^{*}$ has the following physical dimensions: $[F]\,[L]^{-\frac{3+d}{2}}$. Equation (5) describes a straight line with slope $d/2$ in the $\Delta K_{th}$ against $a$ bilogarithmic diagram (Fig. 2).

It can be noted that the power law of Eq. (5) is formally identical to both the empirical relation of Frost [3] and the theoretical law of the Murakami-Endo model [8]. The former assumes an exponent equal to 1/6 for $a$, and is based on experimental data related to crack lengths ranging from 100 to 20000 μm. The latter considers an exponent equal to 1/3, and applies to $\Delta K_{th}$ values determined for crack lengths ranging from 5 to 200 μm. Thus, according to the above authors, the exponent in Eq. (5) can be argued to vary with the crack length.

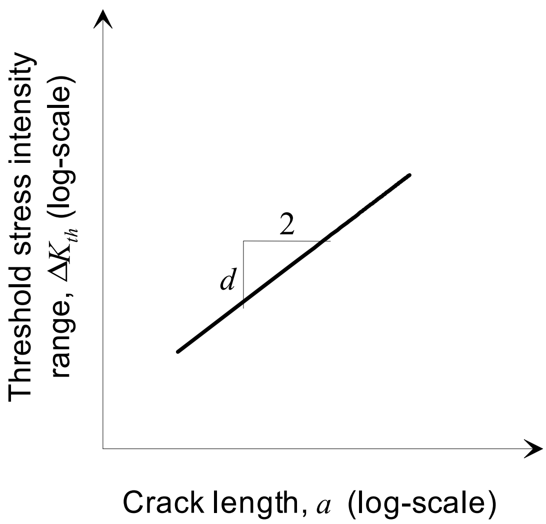



Figure 2. Monofractal scaling law (see Eq. (5)) in the $a - \Delta K_{th}$ bilogarithmic plane.

*Self-Affine Fractal Cracks*

The validity of Eq. (5) is limited by the assumption of a self-similar fractal topology, that is to say, the fractal dimension remains uniform. This implies, for instance, that an infinite threshold stress intensity range would occur with increasing the crack length, which is obviously far from the experimental reality. Hence, in order to describe the $\Delta K_{th}$ against $a$ relationship from small to large values of $a$, we may consider a *multifractal approach*. Accordingly, crack surfaces are treated as self-affine invasive fractal surfaces, so that their fractal dimensional increment $d$ is a function of $a$. The following expression is herein proposed :

$$\Delta K_{th} = \frac{\Delta K_{th}^{\infty}}{\sqrt{1 + \frac{l_0}{a}}} \tag{6}$$

where $\Delta K_{th}^{\infty}$ is the asymptotic threshold stress intensity range for $a \to +\infty$, and $l_0$ is a characteristic length of the material microstructure. Equation (6) describes the fact that fractality decreases as the crack length $a$ increases, namely as $a$ becomes larger and larger with respect to some characteristic length of the material microstructure. As is shown in Fig. 3, the slope of Eq. (6) tends to $1/2$ for $a \to 0^+$ and to 0 for $a \to +\infty$ in the $\Delta K_{th}$ against $a$ bilogarithmic diagram, that is, Equation (6) implicitly assumes that the fractal increment ranges from $d = 1$ for $a \to 0^+$ to $d = 0$ for $a \to +\infty$. Note that Equation (6) is analogous to the multifractal law proposed in Ref. [16] for fracture energy.

It can be seen that Equation (6) is formally identical to Equation (1) describing the Kitagawa diagram according to the ElHaddad model [6]. The comparison between such equations shows that the parameter $\Delta K_{th}^{\infty}$ can be read as the threshold stress intensity

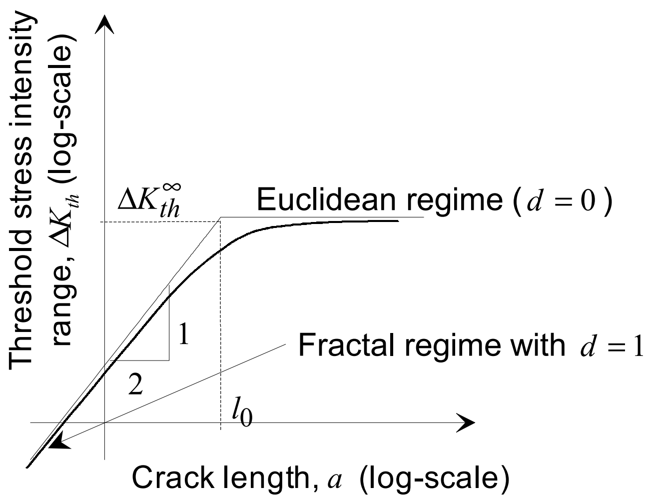



Figure 3. Multifractal scaling law (see Eq. (6)) in the $a - \Delta K_{th}$ bilogarithmic plane.

range $\Delta K_{th0}$ for long cracks (which is crack-size independent), and the material length $l_0$ represents the intrinsic crack length $a_0$.

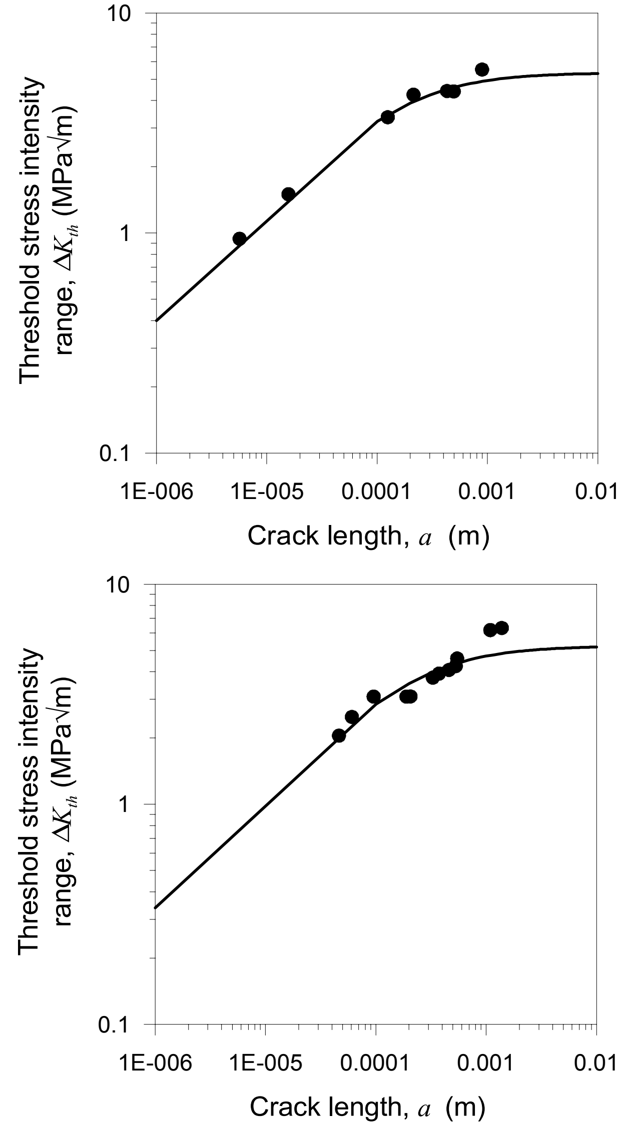
## EXPERIMENTAL APPLICATION

The multifractal law of Eq. (6) is here applied to interpret $\Delta K_{th}$ against $a$ results of some experimental tests carried out by Tanaka and coworkers [7]. The material tested was a ferritic and pearlitic mild steel with the carbon content of 0.20%. The grain size of the ferritic phase was changed by a heat treatment from 7.8 μm (Material A) into 55 μm (Material B). Fatigue tests were conducted on plate specimens at room temperature under fully reversed bending. *K*-decreasing tests were performed to obtain the threshold stress intensity range on plates containing either a centre crack, or a surface crack or a corner crack. The threshold condition was conventionally determined for a crack growth rate equal to $10^{-11}$ m/cycle. The threshold stress intensity range $\Delta K_{th}$ was experimentally evaluated (7 values for Material A, 12 values for Material B) for the crack length ranging from 6 μm to 1383 μm.

The best-fitting $\Delta K_{th}$ against $a$ curves (see Eq. (6)) are shown in Fig. 4 together with the experimental data reported in Ref. [7]. The best-fitting procedure allows us to determine the parameters $\Delta K_{th}^{\infty}$ and $l_0$ of the present model (Table 1). Note that the correlation coefficient $R$ is approaching the unity (corresponding to a perfect correlation) for both materials being examined.

It is self-evident that the tendency of the experimental $\Delta K_{th}$ against $a$ data can be well described also according to ElHaddad model [6], knowing the two parameters $\Delta K_{th0}$ and $a_0$ (see Eq. (1), and Eq. (6) with $\Delta K_{th}^{\infty} = \Delta K_{th0}$ and $l_0 = a_0$). Note that, in the tests by Tanaka and coworkers [7], the value of $\Delta K_{th0}$ was experimentally determined, while the value of $a_0$ (see Table 1) was obtained from Eq. (2) by considering the fatigue limit for smooth specimens ($\Delta\sigma_{th0}$) computed through an empirical expression depending on the grain size.

Therefore, the reason for determining the parameters $\Delta K_{th}^{\infty}$ and $l_0$ from a fitting of $\Delta K_{th}$ against $a$ data is to show a general way of application of the proposed multifractal law, without knowing *a priori* the physical meaning of $\Delta K_{th}^{\infty}$ and $l_0$. Such a meaning is then revealed by comparing Eq. (6) with Eq. (1). In other words, Equation (6) does not provide a new expression for the threshold stress intensity range as a function of the crack length, but it only demonstrates that the relationship describing the Kitagawa diagram can be obtained following a non-conventional approach based on the (multi)fractal geometry.




(a)

(b)

Figure 4. Theoretical curve (see Eq. (6)) and experimental data [7] in the $a-\Delta K_{th}$ bilogarithmic plane: (a) Material A (small grain size); (b) Material B (large grain size).

Table 1. Experimental results [7] and theoretical values computed according to the present model (see Eq. (6)) and the ElHaddad model [6] (see Eq. (1)).

| *Material* | *Grain size* *(μm)* | $\Delta\sigma_{th0}$ *(MPa)* | $\Delta K_{th0}$ *(MPa√m)* | $\Delta K_{th}^{\infty}$ *(MPa√m)* | $l_0$ *(μm)* | $R$ | $a_0$ *(μm)* |
|---|---|---|---|---|---|---|---|
| A | 7.8 | 235 | 5.21 | 5.35 | 179 | 0.999 | 156 |
| B | 55 | 163 | 6.20 | 5.24 | 239 | 0.926 | 461 |

## CONCLUSIONS

Some fractal geometry concepts are exploited to describe the topology of the fracture surfaces. In particular, treating a crack surface as a *self-similar* invasive fractal set, which is characterized by a *uniform* fractal (*monofractal*) dimension, a renormalized (scale-invariant) threshold stress intensity range is firstly defined. This implies a power-type expression between the threshold stress intensity range and the crack length.

Then, modelling the crack surface as a *self-affine* invasive fractal set, which is characterized by a *non-uniform* fractal (*multifractal*) dimension, a general relationship between the threshold stress intensity range and the crack length is proposed. It is shown that such a relationship is formally identical to that of the ElHaddad model [6]. Hence, the present investigation offers a new theoretical basis within the framework of the fractal geometry, according to which the Kitagawa diagram can be justified. Finally, some relevant experimental data [7] are analysed to show how to apply the theoretical fractal approach proposed.

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