Complete and Incomplete Self-Similarity in Fatigue Damage

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***ABSTRACT**. There is a wealth of experimental evidence that, in a wide variety of natural stones, repetition of uniaxial compression loading cycles produces a progressive permanent contraction ε_v in the load direction, which represents a significant indicator of damage progress. Self-similarity arguments are applied to analyze possible forms for kinetic equations of fatigue-damage evolution, correlating for fixed test parameters (frequency, temperature, load intervals etc.) the damage indicator "ε_v" with the number "n" of damaging actions, i.e. each complete loading-unloading cycle. Two distinct phases are predicted by simply using dimensional analysis arguments, provided that "n" is considered a dimensional parameter and further invariance with respect to supplementary groups of similarity transformations is assumed. The first phase is indicated by a pseudo-linear dependence of ε_v upon the logarithm of n. The second phase, prior to failure, is instead characterized by a power-law relationship ε_v vs. n. These two phases of the material behavior have distinct peculiarities at both the mesoscopic and microscopic level. Interpolating curves obtained from these deductions are in excellent agreement with experimental results.*

INTRODUCTION AND PRACTICE

The mechanical response of natural stones subjected to cyclic uniaxial compression was considered in a large experimental study [1]. Cylindrical specimens of marble and Serena Sandstone were tested through varying sequences of two-level programmed loading, measuring the strains through gauge-rosettes placed on the lateral surface of the specimens. A typical diagram for a cyclic test on a particular quality of Carrara marble is reported in Figure 1. Here, the average stress σ is plotted as a function of two different components of strain, ε_v and ε_h, measured respectively in vertical and horizontal direction, that is parallel to the direction of loading or at right angle to that. As can be noted, a typical feature of the fatigue response is that repetition of loading cycles produces a progressive accumulation of permanent strain, rather than significant decay in either the elastic modulus or Poisson's ratio. This behavior is common not only to natural but also to artificial conglomerates (concrete), even if in this latter case the decay of elastic modulus usually accompanies the inelastic deformation.

The following reasons, discussed at length in [1], allow inferring that the inelastic aliquot of strain in the direction of loading, ε_v, can be considered a natural macroscopic indicator of the damage evolution. *i*) There is a linear correlation between the energy dissipated in each hysteresis loop and the inelastic increment of ε_v in the same cycle. *ii*) The ultimate contraction in the direction of loading characterizes the material performance: independently of the load-history, failure occurs when such a parameter, sometimes denoted "strain capacity", reaches a certain limit, characteristic of each rock-type. *iii*) Maintaining fixed the loading intervals, the inelastic part of the deformation exhibits a "steady trend", characterized by a pseudo-linear dependence upon the logarithm of the number of cycles *n* undergone by the specimen.

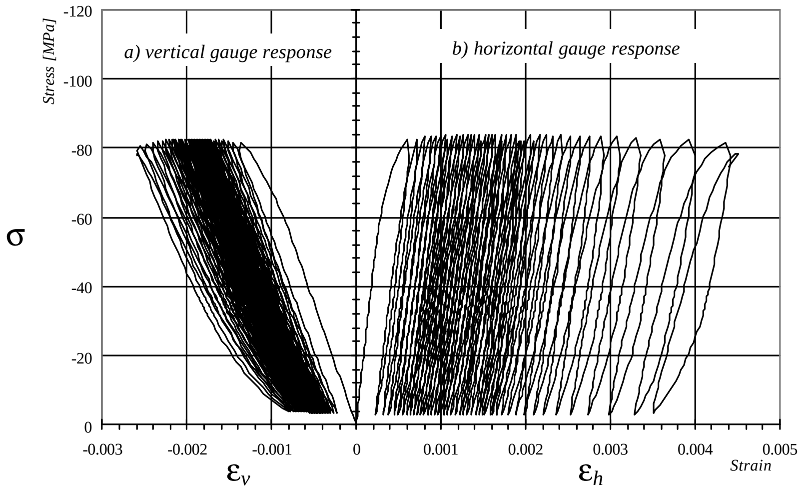
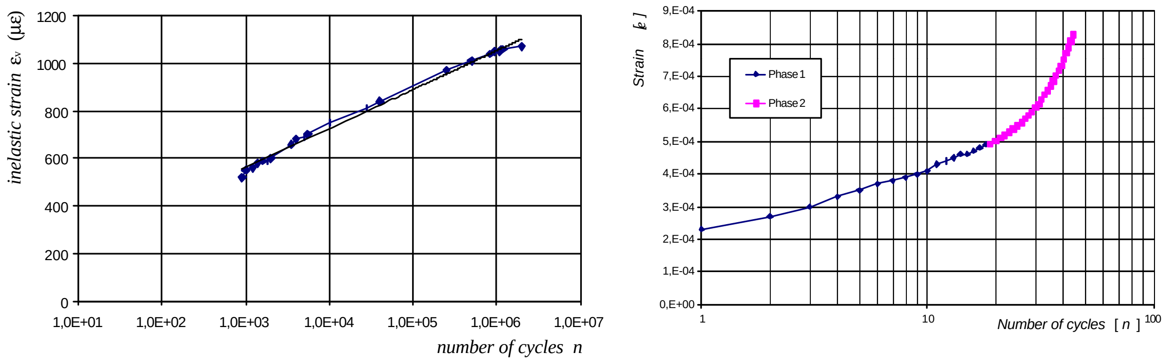


Figure 1. Typical σ-ε relations for cyclic-compression tests on a Carrara Marble.



(*a*) (*b*)

Figure 2. (*a*) Pseudo-linear relationship between inelastic strain and Log(*n*) and (*b*) detail of the final stage, immediately prior to specimen failure. Both graphs are in semi-logarithmic scale.

Figure 2*a* shows, in semi-logarithmic scale, the values of the inelastic part of the deformation ε_v as a function of the number of cycles n that have been necessary to produce it. An interpolation line, whose slope is a significant parameter strictly correlated with the material underlying microstructure [1], evidences the aforementioned pseudo-linear dependence. However, a more careful representation of the latest stage, immediately prior to specimen failure, reveals that such pseudo-linear trend is followed by another stage, in which the permanent contraction marks a sudden pace increase. It has been observed [2] that now there is still a linear proportionality between the energy dissipated in each cycle and the inelastic increment of ε_v in the same cycle, but the increment of vertical strain *per* unit of dissipated energy is less than in the preceding pseudo-linear stage.

The aim of this paper is to discuss these two different "phases" of the fatigue response of natural stones and interpret their evolution with ongoing load cycles from the elementary point of view of dimensional analysis. In particular, our main concern here is to analyze possible forms for a kinetic (evolution) equation correlating ε_v with cycle number n, discussing in particular its self-similar solutions.

DIMENSIONAL ANALYSIS, TRANSFORMATION GROUPS AND SELF-SIMILARITIES

The main ideas for this program are contained in Barenblatt's outstanding book on scaling and self-similarity [3]. Dimensional-analysis-motivated scaling-laws reveal the fundamental property of self-similarity of natural phenomena, i.e. their repeating in time and/or space, which may suggest important simplifications in understanding complex processes and interpret experimental results. A formal consequence of similarity theory is the well-known Buckingam's Π-theorem, which can be stated in the following form. Let the governed parameter a, i.e., the parameter to be determined in the study, be a function of $k+m$ governing parameters, i.e. $a = F\left(a_1, a_2, ..., a_k, b_1, b_2, ..., b_m\right)$. If only the first k parameters $a_1,\ldots,a_k$ have independent dimensions, the obvious fact that physical laws are independent of the choice of basic dimensional scales implies that such a relationship must be invariant with respect to the similarity transformation

$$a_1' = A_1 a_1,\; a_2' = A_2 a_2,\; ....\;,\; a_k' = A_k a_k, \tag{1}$$

corresponding to transition to a different system of units of measurement. Stated more fundamentally, the Π-theorem is a simple consequence of the *covariance principle*: the relations must be invariant with respect to the transformation group (1). Then, it can be proved that the dimensions of the remaining $b_1,\ldots,b_m$ can be expressed as products of powers of the dimensions of $a_1,\ldots,a_k$, that is, $[b_i] = [a_1]^{p_i}\ldots [a_k]^{r_i}$, so that the function $a = F\left(a_1, a_2, ..., a_k, b_1, b_2, ..., b_m\right)$ can always be expressed in the form

$$\Pi = \Phi(\Pi_1, \Pi_2, ...\Pi_m) \quad , \tag{2}$$

where Π, Π_1, …., Π_m, are particular dimensionless combinations of the type

$$\Pi = \frac{a}{a_1^p \dots a_k^r} \quad , \quad \Pi_i = \frac{b_i}{a_1^{p_i} \dots a_k^{r_i}} \quad , \quad i = 1, ..., m \, . \tag{3}$$

It can turn out, however, that there exists a broader group of transformations with respect to which the formulation of the considered problem is invariant, although this similarity is not implied by dimensional analysis. If such a group is, for example,

$$a_1' = a_1 \, , \; .... \; , a_k' = a_k \, , \, b_1' = b_1 , \; .... \; , b_{m-1}' = b_{m-1} \, , b_m' = B b_m , \; a' = a \, , \tag{4}$$

then, it can be demonstrated that the number of arguments of function Φ in (2) should be reduced by the number of varying parameters of the supplementary group, i.e. by one for the case (4). From the viewpoint of intermediate-asympotics [3], this is somehow equivalent to assume that in some physically significant range of its variations, the parameter Π_m in (3) is either very small or very large, and that the function Φ approaches a finite non-zero limit when $|\Pi_m| \to 0$ or $|\Pi_m| \to \infty$. This condition is usually referred to as *complete self-similarity* or self-similarity of the first kind.

Among all the additional possible groups of transformations, a special and very important place belongs to the *renormalization group*

$$a_1' = a_1 \, , \; .... \; , a_k' = a_k \, , \, b_1' = B^{\alpha_1} b_1 , \; .... \; , b_{m-1}' = B^{\alpha_{m-1}} b_{m-1} \, , b_m' = B b_m , \; a' = B^{\alpha_m} a \; , \tag{5}$$

where $\alpha_1, \dots, \alpha_m$ are real or complex numbers. If (5) holds, Φ admits the representation

$$\Pi = (\Pi_m)^{\alpha_m} \; \Phi\left(\frac{\Pi_1}{(\Pi_m)^{\alpha_1}} , \; .... \; , \frac{\Pi_{m-1}}{(\Pi_m)^{\alpha_{m-1}}} \right), \tag{6}$$

so that Π_m remains significant however small or large it may be. This situation is referred to as *incomplete self-similarity*, or self-similarity of the second kind.

In general, possible invariance with respect to groups wider than those indicated by dimensional analysis is suggested by the mathematical formulation of the problem, i.e. the corresponding equations are invariant with respect to transformations of the type (4) or (5). However, even when there is no sound mathematical model, the invariance can be suggested by physical considerations only [3].

Our interpretation of fatigue-damage experiments on natural stones begins with the search for a physical law correlating a macroscopic indicator of damage with the amount of damaging actions undergone by specimen. From the considerations set forth in the Introduction, we surmise that the inelastic increment in axial contraction in the direction of loading, $\Delta\varepsilon_v$, represents a macroscopic measure of the damage produced in the n^{th} cycle. Moreover, if the total number of cycles is so high that n can be considered a continuous parameter, we may assume that the increment of inelastic deformation *per* cycle can be represented as the derivative of a regular function $\varepsilon_v = \varepsilon_v(n)$ with respect to the variable n. Therefore, we can suppose that the quantity $\Delta\varepsilon_v \cong d\varepsilon_v/dn$, calculated at cycle n, is representative of the damage occurring in the same cycle.

On the other hand, we conjecture that the elementary degrading event is represented by a complete loading-unloading cycle. Thus, the number of cycles n also indicates the amount of degrading action the specimen has undergone, at least as long as the other conditions (temperature, frequency, load intervals *etc*.) are kept constant throughout the test. In the simplest case, a theory can be conceived of whereby the minimum set of governing parameters is formed solely by the two quantities ε_v and n. However, the experimental results suggest that there is a transition between two different in type damage evolution, revealed by the two different trends for the ε_v vs. n curves of Figure 2*b*. Consequently, a third variable n_0 should be introduced, indicating a certain cycle number, which represents a *characteristic time scale* in the fatigue evolution and marks the transition from one type of behavior to the other.

Therefore, we surmise that the damage evolution is described by a kinetic ("evolution") equation of the type

$$\frac{d\varepsilon_v}{dn} = f\left(\varepsilon_v, n, n_0\right) \,, \tag{7}$$

where $d\varepsilon_v/dn$ is the governed parameter and ε_v, n and n_0 the governing quantities.

In order to find the form of (7), dimensional analysis is applied first. It is clear that ε_v, being the ratio of two lengths is dimensionless, i.e., $[\varepsilon_v]=1$. Also the parameter n may appear dimensionless, but indeed it holds a deeper physical significance: it represents the number of elementary damaging events the specimens has undergone, each event corresponding to an entire loading-unloading cycle. Other conditions being equal, one complete loading-unloading cycle represents the "*quantum*" of degrading action and the increase of n marks the the evolution of damage in time. Consequently, from a practical point of view, we treat n and n_0 as dimensional parameters, whose dimension will be conventionally indicated with $[N]$. Therefore, $d\varepsilon_v/dn$ has dimensions $[d\varepsilon_v/dn]=[N]^{-1}$. We observe, in passing, that "numbers with dimension" have sometimes being introduced in similarity analysis. Just to mention one example, in [4] dimensional analysis is successfully applied to model the performance of rowing boats accommodating n oarsmen, by considering n (the number of oarsmen) as a dimensional quantity.

We then consider two different stages in the damage evolution.

Stage 1 ($n \ll n_0$). At this stage, n is still far from the transition point marked by the parameter n_0, so that influence of n_0 on (7) is practically negligible. In other words, (7) is supposed to be invariant with respect to the auxiliary similarity transformation of the type (4), characterized by $n_0'=N_0\, n_0$, for arbitrary N_0. From the point of view of dimensional analysis, this is equivalent to state that n and n_0 have *independent* dimensions. A direct use of Π-theorem thus gives

$$n\frac{d\varepsilon_v}{dn} = f_1(\varepsilon_v) \ . \tag{8}$$

In these conditions an intermediate-asymptotic regime may be achieved when the cracks have started to propagate but the material is relatively undamaged and sufficiently far from rupture. This regime is characterized by the invariance with respect to an additional group of similarity transformations of the type (4), with $\varepsilon_v' = E\,\varepsilon_v$ for arbitrary E. Assuming complete similarity in ε_v, damage evolves according to

$$\frac{d\varepsilon_v}{dn} = \frac{A}{n} \quad \Rightarrow \quad \varepsilon_v = A\ln n + C_1 \tag{9}$$

corresponding to the linear semi-logarithmic branch of figures 2.

Stage 2 ($n \cong n_0$). Now the damage history is approaching the transition point n_0, representing an intrinsic critical threshold. Since n and n_0 are comparable, we surmise that (7) is invariant with respect to an auxiliary similarity transformation of the type (4), characterized by $n_0' = N\,n_0$ and $n' = N\,n$ for arbitrary N. This is equivalent to assume that n and n_0 share the same dimensions. Π-theorem gives a condition of the type

$$n_0\frac{d\varepsilon_v}{dn} = f_2\left(\varepsilon_v, n/n_0\right) \ . \tag{10}$$

In general, we see from experiments that n_0 is quite high ($\approx 10^6$ for reasonable load limits), while the duration of the whole second stage is much smaller. Then, the ratio n/n_0 remains sensibly equal to one at this stage. The hypothesis that $f_2(\varepsilon_v,\cdot)$ approaches a definite limit as $n/n_0 \to 1$ is equivalent to a condition of complete similarity in the parameter n/n_0. Assumption of complete similarity in ε_v as well, would mean that the damage produced in each cycle is constant, but this would give a linear relationship ε_v-n that is not matched by the experiments. But if there were an intermediate asymptotic regime characterized by incomplete similarity in ε_v, (10) would become

$$\frac{d\varepsilon_v}{dn} = \frac{B}{n_0}(\varepsilon_v)^\alpha , \tag{11}$$

where α is an undetermined exponent which, as usual in incomplete-similar phenomena, cannot be deduced from covariance principles only [3], but has to be calibrated from the experimental data. Direct integration of this equation gives

$$\varepsilon_v = \left(B(1-\alpha)\,n/n_0 + C_2\right)^{\frac{1}{1-\alpha}} = \left(B^* n + C_2\right)^{\beta}, \qquad (12)$$

that is, ε_v exhibit a power-law dependence upon n.

The undetermined constants (A, C_1, B^*, C_2, β) that appear in (9) and (12) can be calibrated from the experimental data using, for example, the least-square-method approximation.

COMPARISON WITH EXPERIMENTS AND CONCLUSIONS

We tentatively try to interpolate experimental data through a relationship of the type

$$\varepsilon_v = A\ln(n) + C + D\, n^B, \qquad (13)$$

qualitatively comprehending (9) and (12). This represents a continuous transition from the first, self-similar, phase to the second stage, distinguished by incomplete similarity. In fact, since we will find *a posteriori* that $|A|$ $|D|$, for small n the dominant term in (13) is the logarithmic term, i.e. $|A\ln(n)|$ $|Dn^B|$ whereas, for sufficiently large n, $|A\ln(n)|$ $|Dn^B|$. The close approximation that can be obtained through (13) is clearly evident from inspection of Figure 3, which represent, now on a linear scale (not semi-logarithmic), the same data as in Figure 2*b* and the corresponding interpolation curve, deduced from (13). Parameters A, B, C and D, whose values are shown in the graph, have been calibrated using the "method of least squares".

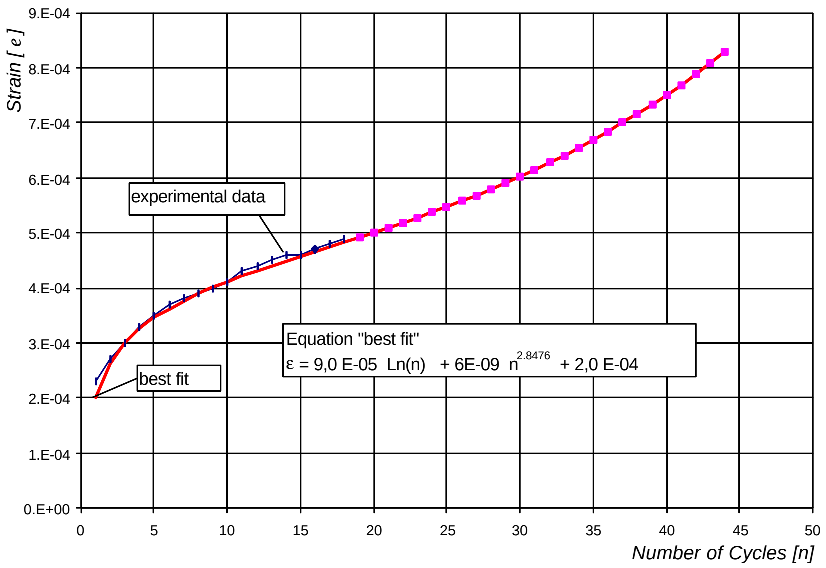


Figure 3. Example of interpolation using relation (13). Same data of Figure 2*b*.

The two aforementioned phases of the material response have also a justification at the microstructural level. The first phase, characterized by the linear semi-logarithmic branch of figures 2, was interpreted in [5] through a simple model based on statistical mechanical considerations. At this stage, fatigue damage produces material slipping at planes inclined approximately 45° with respect to the loading direction. Such slippage depends upon the local value of the shear stress, strongly affected by the stress concentrations inevitably present in the material. These singularities, due to random events such as micro-cracks, micro-inclusions etc., may be represented statistically. Establishing a balance between the chance that the shear stress in any given layer reaches the limit value, and the statistical distribution of the strengths of all possible slip layers, the expected semi-logarithmic linear dependence is confirmed.

Thus, the first phase is characterized by the *diffuse* formation of a network of shear-induced microcracks, nucleated almost independently one another. The beginning of the second stage, when (11) holds, is instead characterized by strain localization, due to the opening of a dominant crack or group of dominant cracks. Let a represent the length of a representative dominant crack, presumably parallel to the maximum shear direction. Its propagation increases the specimen contraction ε_v and we may surmise that there is a linear proportionality between da/dn and $d\varepsilon_v/dn$, i.e. $a \cong h\ \varepsilon_v$. But maintaining fixed the load limits, the variations of the stress intensity factor ΔK depends upon the crack length a. In particular $\Delta K \propto (a)^{1/2}$ or, because of the aforementioned proportionality between a and ε_v, $\Delta K \propto (\varepsilon_v)^{1/2}$. Consequently, supposing that cracks propagate according to Paris-Erdogan law, one finds that

$$\frac{da}{dn} \propto (\Delta K)^m \Rightarrow \frac{d\varepsilon_v}{dn} \propto (\varepsilon_v)^{m/2}, \tag{14}$$

whose form clearly coincides with (11).

SEM pictures of loaded specimen [1-2] have confirmed the gradual appearance, in a first stage of the test, of a diffuse network of microcracks, usually organized in slip planes, which eventually coalesce in dominant cracks that propagate up to failure. Treatment of experimental data from more than one hundred tests on three different qualities of marble [2] and on Serena sandstone confirms that the interpolating curves obtained from (14) are in excellent agreement with the experimental results.

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