J-Integral Applied to Sharp V-Shaped Notches

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ABSTRACT. *In the present paper it has been discussed the physical meaning of J-integral when it is applied to sharply V-notched components by calculating it along a circular path (J_V). Consider a Cartesian reference frame having the x axis parallel to the notch bisector, J_V, for a given circular path, is proportional to the energy release rate of a virtual crack having length equal to the path radius and emanating from the tip of a V-notch. Analytical and numerical results have been performed for linear elastic materials. Finally, as an engineering application, fatigue analysis of welded joints made by using the J_V parameter has been discussed without taking into account the real path direction.*

INTRODUCTION

It is well known that failure analyses can be performed in two-dimensional brittle components even by using the J-integral parameter [1]: the critical value of the applied load is reached when J-integral equals a critical value depending on the material. The most important advantage given by this approach is to overcome asymptotic analyses by using a line integral contour along an arbitrary path. Doing so, the relevant Stress Intensity Factors, K_I and K_{II}, can be calculated by FE analyses obtaining a high degree of accuracy, even by using course meshes. Another J-integral's peculiarity is that it is possible to extend its use to non-linear situations. In fact, when the amount of the ligament is large compared to the specimen width, and the HRR-Stress field is dominant, J supplies a measure of the intensity of the entire elastoplastic stress-strain fields near to the crack tip.

Real components present complex geometries having different kinds of stress concentrators, which include sharp V-notches (for instance, arc welding, shaft, bonded joints, etc.). For this reason, it could be interesting to generalise the J-integral's use, by applying it to generic V-notches, to perform both notch and failure analyses. This kind of approach has been widely discussed in Refs [2,3]. In order to avoid confusion, the J-integral value can be indicated as J_V, when it is applied to V-notched components. The J_V analytical expression has been previously formalised as a function of the integration path both for linear-elastic and elasto-plastic materials [2]. However, in particular situations, it is possible to obtain two path-independent integrals, named J_{L1} and J_{L2}, for mode I and mode II, respectively, by employing an "ad hoc" adjustment of the classical J-integral expression [2,3]. Moreover, it is interesting to highlight that J_V can be even

applied to rounded V-notches making explicit the bridging between the actual peak stress and the corresponding Notch Stress Intensity Factors (NSIF) [4].

By using different theories [1,5], the physical meaning of J-integral can be easily understood in the presence of cracks. Unfortunately, its meaning has not been still clarified when it is applied to sharp V-shaped notches. In the present paper it will be addressed the problem of proposing a new physical interpretation for this parameter. Finally, it can be highlighted that this approach could be even used in the fracture mechanics field in the presence of sharp V-notches when the exact crack path direction is not fundamental for estimating the total fatigue life.

ANALITICAL BACKGROUND

The J-integral parameter was defined by Rice [1] as a line integral between two points, A and B, of a plate subjected to a two-dimensional deformation field, as:

$$J = \int_{\Gamma} \left(W \, dy - T \cdot \frac{\partial u}{\partial x} \, d\Gamma \right) \tag{1}$$

where Γ is a curve surrounding the notch tip and the integral is evaluated in a counter-clockwise sense. W is the strain-energy density, $\mathbf{T}$ is the traction vector defined according to the outward normal along Γ and $\mathbf{u}$ the displacement vector. If the A and B points are taken on the two opposite faces of a crack, J-integral gives a constant value. When the material behaviour can be considered as linear-elastic, under mixed mode loadings (mode I and mode II) J-integral assumes the following well known expression (the x-x axis of the Cartesian frame of reference must be parallel to the crack faces):

$$J = J_I + J_{II} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} \tag{2}$$

where K_I and K_{II} are the Stress Intensity Factors and E' is equal either to the Young modulus, E, under plane stress or to $E/(1-\nu)$ under plane strain.

It is also well known that J-integral represents the rate of decrease per unit thickness, t, of the potential energy Π with respect to the crack size [1]

$$J = -\frac{1}{t} \frac{d\Pi}{da} \tag{3}$$

On the other hand, when only configurational forces are accounted for, J-integral provides the balance around the crack tip, as soon as the free energy is taken into account in the configurational traction [5,6].

Recent results [2, 3] indicate that, under mixed mode loadings, J_V calculated in a circular path (i.e. $r_1=r_2$ in Fig. 1a), can be expressed as follows

$$J_V = J_{V1} + J_{V2} = \ r^{2\lambda_1 - 1} J_{L,1} + r^{2\lambda_2 - 1} J_{L,2} \ ; \qquad J_{L,i} = \frac{J_{Vi}}{r^{2\lambda_i - 1}} = \frac{\bar{J}_i}{E'} (K_i^N)^2 \qquad (4)$$

where J_{Li} are two path independent integrals linked to the relevant NSIFs, K_i^N, and to Williams' eigenvalues, λ_i [7], while $\bar{J}_i$ depends just on the notch opening angle, 2α. Table 1 summarises the aforementioned λ_i and $\bar{J}_i$ values for selected 2α values.
Note that when $2\alpha=0$, Eq. (4) coincides with Eq. (2) and the path dependence of J_V from r disappears.

Table 1. Parameter for J_V calculation

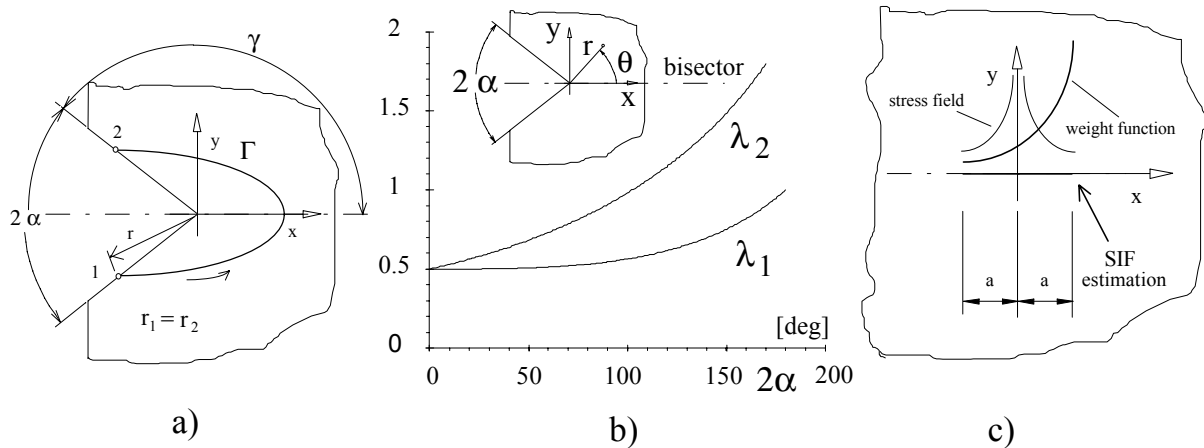| 2α	λ_1	λ_2	$\bar{J}_1$	$\bar{J}_2$	A_1	A_2
0	0.500	0.500	1	1	-	-
$\pi/6$	0.501	0.598	0.993	0.801	1.67	1.45
$\pi/3$	0.512	0.731	0.943	0.583	1.64	1.25
$\pi/2$	0.544	0.909	0.812	0.389	1.56	1.07
$2\pi/3$	0.616	1.149	0.597	0.236	1.42	0.909
$3\pi/4$	0.674	1.302	0.474	0.176	1.33	0.836

Figure 1. a: integral path surrounding the notch tip;
b: Williams' eigenvalues [7];
c: weight function approach.

SIF FOR CRACKS NUCLEATED FROM SHARP V- NOTCHES

The SIF calculation for a crack emanating from a triangular notch has been performed by Hasebe and Iida [8] employing rotational mapping functions. According to Ref. [9] as well as to Gross and Mendelson's definition for the NSIF parameter, the mode I SIF value of a crack having a length, a, and emanating from a V-notch, takes the form:

$$K_I \propto K_I^N \, a^{\lambda_1 - 1} \sqrt{a \pi} \tag{5}$$

Note that [11], due to nature of the physical dimension of K_1^N, the K_I dependence from the crack length is different from 0.5 as in the case of non-singular stress fields in the neighbourhoods of the crack initiation point.

In order to generalise Eq. (5) to mode II loadings, the weight function technique can be used. If the weight function is known for the actual geometry, the SIF may be expressed as follows:

$$K_I = \int_{crack} w \, \sigma \, dx \tag{6}$$

where w is the weight function depending only on the geometry and σ is either the normal stress for mode I or the shear stress for mode II. The advantage of this approach is that stresses have been computed on the un-cracked body along the crack path. At this point, for the sake of simplicity, it can be considered just the case of a $2a$-length crack in an infinite body subjected to mode I and II loadings. Approximately, SIFs for lateral cracks of length a in semi-infinite bodies (differing from the former case) can be calculated by using the Koiter's coefficient equal to 1.1215 (for different specific weight functions one can see other references, see for example Ref. [9]). For small crack in large plates, both mode I and II weight functions can be written in the following form:

$$w = \sqrt{\frac{1}{\pi a}} \, \sqrt{\frac{a+x}{a-x}} \tag{7}$$

Considering that singular stress fields occur at sharp notches, the stress component according to Williams [7] can be expressed, in a polar coordinate system, as:

$$\sigma_{jk} = \beta_{jk,1} \, K_1^N \, r^{\lambda_1 - 1} + \beta_{jk,2} \, K_2^N \, r^{\lambda_2 - 1} \qquad j,k = r, \theta \tag{8}$$

where β_{jk} is a coefficient depending on the 2α notch opening angle and on the actual θ direction, as demonstrated in Refs [7,10] (see Fig. 1b). In the case of $\theta=0$, $\beta = \beta_{\theta\theta} = \beta_{r\theta} = 1/\sqrt{2\pi} = 0.399$.

According to Eqs (6-8), the stress intensity factor for a crack embedded in a singular stress field, becomes (see Fig. 1c):

$$K_i = \int_{-a}^{a} \beta \ K_i^N \ |x|^{\lambda_i - 1} \ \sqrt{\frac{1}{\pi a}} \ \sqrt{\frac{a+x}{a-x}} \ dx \qquad (9)$$

A closed form solution of Eq. (9) for real value of λ_i cannot be expressed in terms of an elementary function and just a numerical integration can be performed to calculate it. On the contrary, for integer values of λ, integral (9) can be obtained in analytic form. At this point, it could be convenient to numerically solve Eq. (9), and then, on the basis of dimensional analyses, write the SIF value as:

$$K_i = A_i \ \beta \ K_i^N \ a^{\lambda_i - 1} \ \sqrt{\pi a} \qquad (10)$$

being A_i the integration parameters reported in Table 1. Analogous results were obtained in Ref. [11] only for mode I loadings by taking into account Albrecht-Yamada's simplified approach, applied to a lateral crack emanating from a sharp notch.

An important peculiarity of Eq. (10) is that the Stress Intensity Factor of a crack at the apex of a V-notch depends on its dimension a with a power exponent equal to (λ-1/2). Furthermore, according to Eq. (2), for any path surrounding the crack tip, the J-integral turns out to be:

$$J_i = \frac{\left(A_i^2 \ \beta \ K_i^N \ a^{\lambda_i - 1} \ \sqrt{\pi a}\right)^2}{E'} = \frac{\pi \ A_i^2 \ \beta^2 \ (K_i^N)^2 \ a^{2\lambda_i - 1}}{E'} \qquad \text{for} \ \ i = 1, 2 \quad (11)$$

A BRIDGING BETWEEN J_V AND SIF FOR AN EMBEDDED CRACK

Up to this point, our discussion has been managed to separately see the J_V application to a sharp V-notch without crack and the J-integral application to cracks ahead of the tip of V-notches.

The similarity between Eq. (4) and Eq. (11) is obvious and suggests a correlation between J_V and J-integral. Therefore, for a given V-notch and an arbitrary circular path R (Fig. 2), the ratio between J_V and J of a virtual crack is constant:

$$\frac{J_i \big|_{a=R}}{J_{Vi} \big|_R} = \text{constant} \qquad (12)$$

Analysing the problem from a different point of view, J_V can be considered as the energy release rate of a virtual crack having a length equal to the path radius R:

$$J_V\big|_R \propto \frac{d\,\Pi}{da}\bigg|_{a=R} \tag{13}$$

At this point, a useful correspondence may be drawn among K_i^N, J_L, J_V and J. Under mixed mode loadings, for a given V-notch, $J_{L,i}$ is to K_i^N as J_V is to J-integral of a virtual embedded crack along the bisector.
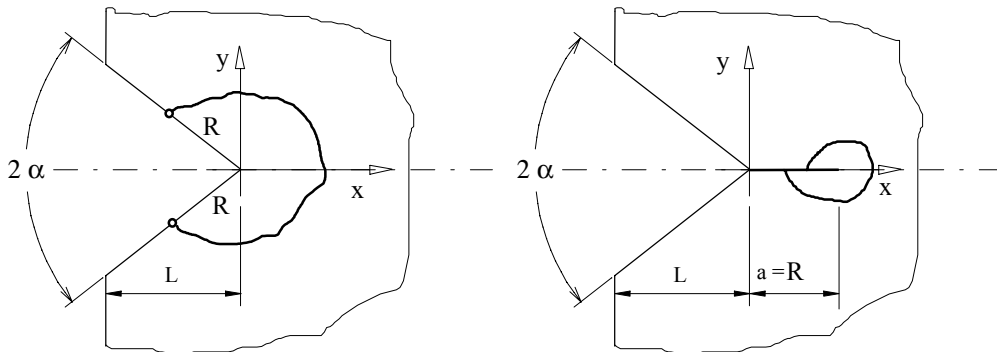

Figure 2. Frame of reference.

The numerical check of Eqs (4) and (12) performed by FE method is reported in Table 2. The listed results are relative to the specimens showed in Fig. 3. In order to minimise the computational errors, using the ANSYS software it has been performed accurate mesh with quad 8 elements. The 2α opening angle was equal to 60° and 135°, respectively. Note that mode II is singular only in the former case.

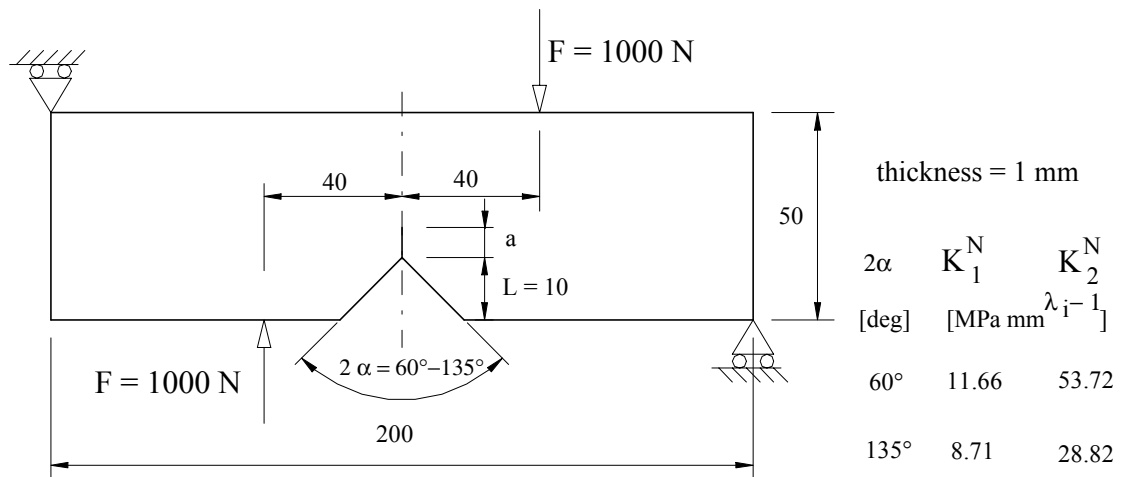| 2α	K_1^N	K_2^N
[deg]	[MPa mm$^{\lambda_i-1}$]	
60°	11.66	53.72
135°	8.71	28.82

Figure 3. Specimens used for FEAs (dimension in mm).

Table 2. Numerical evaluation of J_V and comparison between J_V and J-integral of a crack performed by means of FEAs in samples having the shape sketched in Fig. 3.

a/L or R/L for $2\alpha=60°$	$\dfrac{J_{V,FEM}\big|_R}{J_{V,eq.4}\big|_R}$ a=0	$\dfrac{J_1\big|_{a=R}}{J_{V1}\big|_R}$	$\dfrac{J_2\big|_{a=R}}{J_{V2}\big|_R}$	a/L or R/L for $2\alpha=135°$	$\dfrac{J_{V,FEM}\big|_R}{J_{V,eq.4}\big|_R}$ a=0	$\dfrac{J_1\big|_{a=R}}{J_{V1}\big|_R}$	$\dfrac{J_2\big|_{a=R}}{J_{V2}\big|_R}$
0.00924	0.999	1.09	1.31	0.0042	0.991	1.77	2.28
0.0462	0.997	1.08	1.34	0.0209	0.998	1.76	2.09
0.100	0.982	1.13	1.36	0.1045	1.007	1.73	2.15

FATIGUE APPLICATION OF ELASTIC ΔJ_V

If the most of fatigue life is spent to nucleate and propagate a crack in a small structural volume (see Sih [12] for the crack case or Lazzarin and Zambardi for the V-notch case [13]), one could make the assumption that fatigue life assessments can be performed by controlling only the local stress field. The welded structures analysed in Ref. [14] fall in the aforementioned case. In fact, 50-70% of the total fatigue life of cruciform fillet-welded having 25 mm thickness was spent to propagate a crack up to 1 mm and 80-90% up to 3 mm [14]. Figure 4 shows the trend of the elastic ΔJ_{V1} against the total fatigue life of about 180 experimental points obtained by testing steel welded structures previously analyse in terms of NSIFs (for details see Ref. [15]). For these welded structures the fatigue crack path was perpendicular to the main plate ([16,17]) and mixed mode stresses were present at weld toe [15]. Here, the critical path radius R_{cr} was set equal to the unity. This is an arbitrary choice, but the discussion of the right definition of a path radius requires further investigations. However, if we take into account experimental failures at the root, as done in Refs [18,19], they fall into the scatter band in Fig. 4. Furthermore, if one considers the ΔK_{th} threshold values for welded structural steel of about 180 MPa mm$^{0.5}$, as proposed by Raday [20], the relative ΔJ falls into the scatter band at $5 \cdot 10^6$ cycles.

Finally, note that, considering even the contribution due to mode II, the value of the total elastic ΔJ_V is practically coincident with that given by ΔJ_{V1}.

CONCLUSIONS

In the present paper it has been established, under mixed mode loadings, a relationship between the J-integral applied to V-notches (J_V) and the classic J-integral. For any path radius R surrounding the notch tip, J_V is proportional to the J-integral of a virtual embedded crack having a length equal to R.

Additionally, in order to compare fatigue lives of different welded joints, a fictitious critical value of the path radius of 1 mm has been considered. The fatigue life of steel cruciform welded joints, showing failures either at the toe or at the root, fall into the same scatter band without taking into account the real path direction.
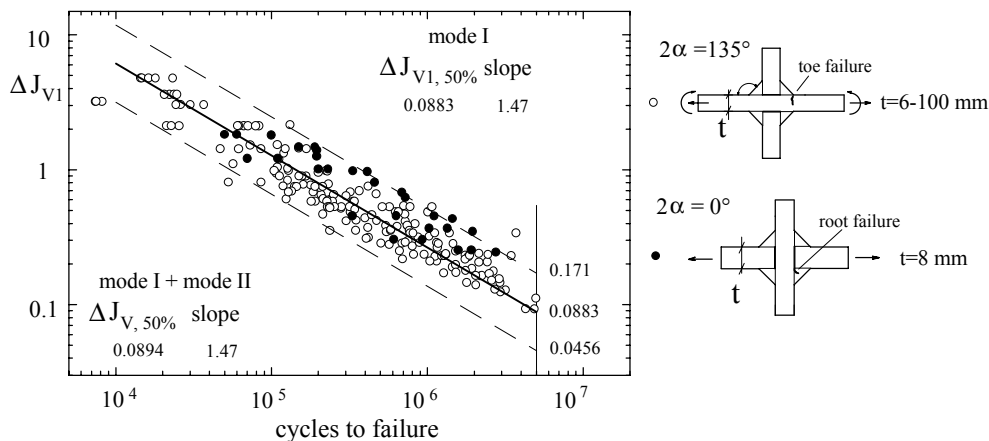
Figure 4. Fatigue strength of steel welded joints as a function of the elastic parameters ΔJ_V. Scatter band related to mean values plus/minus 2 standard deviations ($R_{cr} = 1$ mm).

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