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On the Crack Extension Energy Rate in Elastic–Plastic Bodies

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ABSTRACT After a brief reference to certain aspects of the flow theory of plasticity with work-hardening, the equation is supplied which expresses the work done by internal forces during an arbitrary deformation process. The crack extension energy rate and the J-integral are then defined in a way which is appropriate for elastic–plastic cracked bodies and a number of their properties are established.

Introduction

Preliminaries

Bold-face lower case letters indicate vectors in $\mathbb{R}^2$; upper case letters indicate second-order tensors (that is, linear transformations of $\mathbb{R}^2$ into $\mathbb{R}^2$). Using A to denote a second order tensor, A^{T} is the transpose of A and $A_0 = A - \frac{1}{3}(\mathrm{tr}\ A)I$, with tr being the trace and I, the identity tensor, is the traceless part of A. Sym is the collection of all second-order symmetric tensors and Sym_0 is the collection of all the traceless elements of Sym. Sym can be made into an inner product space by defining $A \cdot B = \mathrm{tr}\,(AB)$, for $A,\ B \in \mathrm{Sym}$. We write $\|A\| = (A \cdot A)^{1/2}$ for the modulus of A.

In the case of hyperelastic materials, the crack extension energy rate G, that is, the rate of energy absorbed at the crack tip during crack extension, is called the energy release rate. In linear elastic cracked solids, in conditions of plane stress or plane strain, G was evaluated, over a period of time, on the basis of the stress intensity factor **(1)** and subsequently by calculating the J-integral, formulated by Rice in 1968 **(2)**, which makes it possible to also assess G in non-linear elastic materials.

If body $\mathscr{B}$ is hyperelastic we put **(3)**

$$G(l) = -\frac{\mathrm{d}}{\mathrm{d}l}\int_{\mathscr{B}} \sigma\ \mathrm{d}a + \int_{\partial\mathscr{B}} T\boldsymbol{\nu} \cdot \boldsymbol{u}'\ \mathrm{d}s \tag{1}$$

where l, σ, T, $\boldsymbol{\nu}$, and $\boldsymbol{u}$ are the crack length, the strain–energy density, the Cauchy stress, the outward unit normal to $\partial\mathscr{B}$ and the displacement field,

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respectively. Moreover, if γ is a path around the tip, that is, a smooth non-intersecting path that begins and ends on the crack and surrounds the tip (Fig. 2), the quantity

$$J(\gamma) = \boldsymbol{e} \cdot \int_\gamma (\sigma I - \nabla \boldsymbol{u}^T T)\boldsymbol{n} \, \mathrm{d}s \tag{2}$$

where $\boldsymbol{e}$, I, and $\boldsymbol{n}$ are the direction of propagation of the crack, the identity tensor, and the outward unit normal on γ, respectively, is the J-integral corresponding to curve γ. The tensor $(\sigma I - \nabla \boldsymbol{u}^T T)$ is called the energy momentum tensor.

If the body is homogeneous and subject to quasi-static deformations in the absence of body forces, we have **(4)**

$$\operatorname{div} (\sigma I - \nabla \boldsymbol{u}^T T) = 0 \tag{3}$$

This result proves to be particularly useful when G needs to be calculated for a two-dimensional body with a straight crack. As is well known **(3)**, in this case we have

$$G = J(\gamma) \tag{4}$$

for each path γ around the tip.

In **(5)** and **(6)** a definition is given of the J-integral for elastic–plastic materials described by a flow theory of plasticity, which uses the work done by internal forces (stress work density) as a substitute for strain–energy density. What is being discussed, then, is the relevance of J defined in this way as a fracture parameter.

The present paper examines elastic–plastic hardening materials described by a flow theory of plasticity which satisfy the classic von Mises criterion. For these materials, we supply the equation which expresses the work $\hat{w}_E(\tau)$ done by internal forces up to time τ during any deformation process $\hat{E}$, as a function of the current value of deformation $\hat{E}(\tau)$, plastic deformation $\hat{E}^p(\tau)$, and the Odqvist parameter $\zeta(\tau)$,

$$\hat{w}_E(\tau) = \hat{w}(\hat{E}(\tau), \hat{E}^p(\tau), \zeta(\tau)) \tag{5}$$

Therefore, we put

$$G(l) = -\frac{\mathrm{d}}{\mathrm{d}l} \int_{\mathscr{B}} w \, \mathrm{d}a + \int_{\partial\mathscr{B}} T\boldsymbol{v} \cdot \boldsymbol{u}' \, \mathrm{d}s \tag{6}$$

for the crack extension energy rate, and we determine conditions under which it is possible to prove the relation

$$G(l) = \lim_{\delta \to 0} \boldsymbol{e} \cdot \int_{\partial\mathscr{D}_\delta} (w\boldsymbol{n} - \nabla \boldsymbol{u}^T T\boldsymbol{n}) \, \mathrm{d}s \tag{7}$$

Subsequently, as in the case of hyperelastic bodies, we put

$$J(\gamma) = \boldsymbol{e} \cdot \int_\gamma (wI - \nabla \boldsymbol{u}^T T)\boldsymbol{n} \, \mathrm{d}s \tag{8}$$

for the J-integral and we prove that

$$G(l) = J(\gamma) - \boldsymbol{e} \cdot \int_{\mathscr{L}} \operatorname{div} (wI - \nabla \boldsymbol{u}^T T) \, \mathrm{d}a \tag{9}$$

where γ is any path around the tip and $\mathscr{L}$ is the intersection between $\mathscr{B}_\zeta$, the region of the body in which plastification has taken place and the region bounded by γ. As is well known, for steady-state growth in ideally-plastic solids the right-hand side of (7) vanishes **(12)**, **(13)**; however, numerical calculation of G for elastic–plastic hardening solids is made considerably easier by knowledge of equation (5).

Lastly, it is proved that, where the deformation process in each point of $\mathscr{B}_\zeta$ is straight and monotonous, we get $\operatorname{div} (wI - \nabla \boldsymbol{u}^T T) = 0$, and from (9) we obtain an equation that is very similar to (4), valid for hyperelastic materials. This is direct proof of the well-known fact that equation (4) also applies to materials described by the deformation theory of plasticity.

In the present paper we confine ourselves to considering two-dimensional bodies; the extension of certain results to three-dimensional problems can be obtained by means of a procedure similar to that followed in **(6)** and **(7)**.

When putting forward hypotheses and deducing properties of the crack extension energy rate, we shall refer continuously to **(3)**, even though in the present paper certain intermediate results proved in **(3)** are, for the sake of simplicity, assumed as hypotheses.

Constitutive hypotheses

In this section, for the reader's convenience, we shall briefly present, in axiomatic form, certain elements of the flow theory of infinitesimal plasticity which, as shown in **(8)**, can be deduced from a general theory of materials with elastic range on the assumption, accepted in the present paper, that the displacement gradient from a fixed reference configuration is small. We shall begin with a number of indispensable definitions.

A deformation process or, more briefly, a history of duration $\bar{\tau}$ is a continuous and continuously piecewise differentiable mapping, defined on the closed real interval $[0, \bar{\tau}]$ with values in Sym,

$$\hat{E}: [0, \bar{\tau}] \to \mathrm{Sym}, \qquad \tau \mapsto \hat{E}(\tau) \tag{10}$$

such that

$$\hat{E}(0) = 0 \tag{11}$$

Value $\hat{E}(\tau)$ at instant τ of a history $\hat{E}$ is interpreted as the infinitesimal deformation, that is, the symmetrical part of the displacement gradient, starting from a fixed reference configuration, in a fixed material point. At each instant τ in which $\hat{E}$ is differentiable, $\dot{E}$ represents the value of the derivative of $\hat{E}$ at instant τ; for each τ for which $\dot{E}$ is discontinuous we shall indicate the right-hand derivative as $\dot{E}$. All deformation processes are thought to begin at some fixed initial state.

The materials being considered here are elastic–plastic isotropic solids whose mechanical response to deformation processes is described by a frame-indifferent and rate-independent constitutive functional. For each history $\hat{E}$ we use $\hat{T}_E(\tau)$ to denote the stress at time τ associated with history $\hat{E}$ by the constitutive functional.

The kind of constitutive response is further specified by the notion of elastic range and plastic history.

Elastic range $\mathscr{E}_E(\tau)$ at time τ corresponding to history $\hat{E}$ is the closure of an arcwise connected open subset of Sym, whose boundary is attainable from interior points only; it contains $\hat{E}(\tau)$ and its points are interpreted as the infinitesimal deformations from the reference configuration to configurations which are elastically accessible from the current configuration.

Plastic history $\hat{E}^p$ corresponding to $\hat{E}$ is the history such that, for each $\tau \in [0, \bar{\tau}]$, $\hat{E}^p(\tau)$ is a traceless tensor, belongs to $\mathscr{E}_E(\tau)$ and corresponds to an unstressed configuration.

It is then supposed there exist two material constants λ and μ such that, if $\hat{E}$ and $\hat{E}^p$ are a history and the corresponding plastic history, respectively, we have, for each $\tau \in [0, \bar{\tau}]$,

$$\hat{T}_E(\tau) = \mathbb{T}[\hat{E}(\tau) - \hat{E}^p(\tau)] = 2\mu(\hat{E}(\tau) - \hat{E}^p(\tau)) + (\lambda \operatorname{tr} \hat{E}(\tau))I \tag{12}$$

Relation (12) reflects the classical hypothesis that the stress response to a purely elastic strain from the unstressed configuration reached after unloading at the current instant τ is both unaffected by the past deformation process and completely determined by $\hat{E}(\tau)$ and $\hat{E}^p(\tau)$.

For each history $\hat{E}$ and for each $\tau \in [0, \bar{\tau}]$

$$\zeta_E(\tau) = \int_0^\tau \|\dot{E}^p(\tau')\| \, d\tau' \tag{13}$$

is the length of the path described up to instant τ by the plastic deformation tensor in Sym_0. ζ is called the *Odqvist parameter.*

In view of the applications we have in mind, we accept the von Mises criterion. That is to say, we suppose that for each history $\hat{E}$ and for each $\tau \in [0, \bar{\tau}]$ the corresponding elastic domain is the cylinder

$$\mathscr{E}_E(\tau) = \{E \in \mathrm{Sym} / \|E_0 - \hat{C}_E(\tau)\| \leqslant \rho(\zeta(\tau))\} \tag{14}$$

where

(i) ρ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable, non-decreasing function that depends on the material but is independent from the history;

(ii) for each history $\hat{E}$, $\hat{C}_E$ is a history which takes its values in Sym_0.

Moreover, in order to take the Bauschinger effect into account, we accept the classic kinematic hardening rule proposed by Melan **(9)**. That is to say, we suppose there exists a non-negative constant η such that for each history $\hat{E}$ and for each $\tau \in [0, \bar{\tau}]$ we have

$$\hat{C}_E(\tau) = (1 + \eta)\hat{E}^p(\tau) \tag{15}$$

In particular, a material for which ρ is a constant function and for which we have $\eta = 0$ is called *ideally plastic.*

The set of constitutive hypotheses is completed by the *flow rule* which states that, when $\dot{E}^p(\tau)$ is different from zero, it is parallel to $\hat{N}_E(\tau)$, the outward unit normal on the elastic range at $\hat{E}(\tau)$,

$$\dot{E}^p(\tau) = \dot{\zeta}_E(\tau)\hat{N}_E(\tau), \qquad \hat{N}_E(\tau) = [\rho(\zeta_E(\tau))]^{-1}(\hat{E}(\tau) - \hat{C}_E(\tau)) \tag{16}$$

As proved in **(8)**, for the evolution of ζ_E we have the following equation

$$\dot{\zeta}_E(\tau) = \begin{cases} 0 \text{ if } \|\hat{E}(\tau)_0 - \hat{C}_E(\tau)\| < \rho(\zeta_E(\tau)) \\ 0 \text{ if } \|\hat{E}(\tau)_0 - \hat{C}_E(\tau)\| = \rho(\zeta_E(\tau)) \text{ and } \hat{N}_E(\tau) \cdot \dot{E}(\tau)_0 \leqslant 0 \\ [1 + \eta + \rho'(\zeta_E(\tau))]^{-1}\hat{N}_E(\tau) \cdot \dot{E}(\tau)_0 \\ \quad \text{if } \|\hat{E}(\tau)_0 - \hat{C}_E(\tau)\| = \rho(\zeta_E(\tau)) \text{ and } \hat{N}_E(\tau) \cdot \dot{E}(\tau)_0 > 0. \end{cases} \tag{17}$$

where we put $\rho' = d\rho/d\zeta$. When one of the first two cases contemplated in the right-hand side of (17) occurs, the material behaves elastically; the third case is known as the *plastic loading condition.*

For each history $\hat{E}$ and for each $\tau \in [0, \bar{\tau}]$, $\hat{T}_E(\tau) \cdot \dot{E}(\tau)$ is the *stress power* and thus the *work done by internal forces* in the deformation process $\hat{E}$ up to time τ, is

$$\hat{w}_E(\tau) = \int_0^\tau \hat{T}_E(\tau') \cdot \dot{E}(\tau') \, d\tau' \tag{18}$$

With the following proposition a relation is proved that, for each history $\hat{E}$ and for each time $\tau \in [0, \bar{\tau}]$, expresses $\hat{w}_E(\tau)$ as a function of $\hat{E}(\tau)$, $\hat{E}^p(\tau)$, and $\zeta(\tau)$ **(10)**.

Proposition 2.1

For each history $\hat{E}$ and for each $\tau \in [0, \bar{\tau}]$ we have

$$\hat{w}_E(\tau) = \tfrac{1}{2}(\hat{E}(\tau) - \hat{E}^p(\tau)) \cdot \mathbb{T}[\hat{E}(\tau) - \hat{E}^p(\tau)] + \mu\eta\|\hat{E}^p(\tau)\|^2 + 2\mu\omega(\zeta_E(\tau)) \tag{19}$$

where ω is the primitive of ρ such that $\omega(0) = 0$.

Proof

Remembering that the plastic deformation is traceless, we deduce from (18) and (12) that

$$\hat{w}_{\mathrm{E}}(\tau) = \int_0^\tau \hat{T}_{\mathrm{E}}(\tau') \cdot \dot{E}(\tau')\,\mathrm{d}\tau' = \int_0^\tau \dot{E}(\tau') \cdot \mathbb{T}[\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')]\,\mathrm{d}\tau'$$

$$= \int_0^\tau \{(\dot{E}(\tau') - \dot{E}^{\mathrm{p}}(\tau')) \cdot \mathbb{T}[\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')] + \dot{E}^{\mathrm{p}}(\tau') \cdot \mathbb{T}[\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')]\}\,\mathrm{d}\tau'$$

$$= \tfrac{1}{2}[(\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')) \cdot \mathbb{T}[\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')]]_0^\tau + \int_0^\tau 2\mu(\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')) \cdot \dot{E}^{\mathrm{p}}(\tau')\,\mathrm{d}\tau'$$

Moreover, from (15), (16), and (17) we obtain

$$\int_0^\tau 2\mu(\hat{E}(\tau') - \hat{E}^{\mathrm{p}}(\tau')) \cdot \dot{E}^{\mathrm{p}}(\tau')\,\mathrm{d}\tau' = \int_0^\tau 2\mu\rho(\zeta_{\mathrm{E}}(\tau'))\hat{N}_{\mathrm{E}}(\tau') \cdot \dot{E}^{\mathrm{p}}(\tau')\,\mathrm{d}\tau'$$

$$+ \int_0^\tau 2\mu\eta\hat{E}^{\mathrm{p}}(\tau') \cdot \dot{E}^{\mathrm{p}}(\tau')\,\mathrm{d}\tau' = [2\mu\omega(\zeta_{\mathrm{E}}(\tau')) + \mu\eta\|\hat{E}^{\mathrm{p}}(\tau')\|^2]_0^\tau$$

The desired result now follows from the fact that we have $\hat{E}(0) = \hat{E}^{\mathrm{p}}(0) = 0$, $\omega(\zeta_{\mathrm{E}}(0)) = 0$.

Preliminary assumptions

In this section we propose to extend certain results of fracture mechanics, which are well known in the case of hyperelastic materials, to elastic–plastic problems **(3)**.

Let $\mathscr{B}$ be a regular homogeneous two-dimensional elastic–plastic body whose mechanical response to deformation processes is described by the constitutive equations (12)–(17). Let us identify $\mathscr{B}$ with the particular region of $\mathbb{R}^2$ occupied by the body in the reference configuration. Let us consider a motion of $\mathscr{B}$ which takes place in the time interval $[0, \bar{\tau}]$, and for each $x \in \mathscr{B}$ let us use $\hat{E}_x$ and $\hat{E}_x^{\mathrm{p}}$ to indicate the deformation process and the corresponding plastic deformation process at point x, respectively.

Let us suppose $\mathscr{B}$ contains an edge crack, represented at every instant τ by the image of a smooth non-intersecting curve,

$$\hat{\chi}_l\colon [0, l] \to \mathscr{B}\alpha \mapsto \hat{\chi}_l(\alpha) \tag{20}$$

parametrized by arc length α (Fig. 1). The length $l = l(\tau)$ of curve $\hat{\chi}_l$ is a non-decreasing function of time τ during the motion of $\mathscr{B}$, and if l_2 is greater than l_1, $\hat{\chi}_{l_2}$ is a continuation of $\hat{\chi}_{l_1}$.

Let us use $\mathscr{C}(l)$ to indicate the image of path $\hat{\chi}_l$,

$$\mathscr{C}(l) = \{x \in \mathscr{B}/x = \hat{\chi}_l(\alpha)\alpha \in [0, l]\} \tag{21}$$

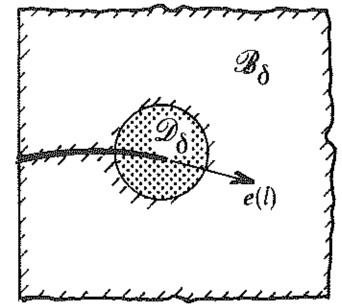

Fig 1 Edge crack

$\hat{\chi}_l(0)$ and $\hat{\chi}_l(l)$ represent the intersection of the crack with the boundary $\partial\mathscr{B}$ of $\mathscr{B}$ and the tip of the crack, respectively, while

$$e(\alpha) = \mathrm{d}\hat{\chi}_l/\mathrm{d}\alpha \tag{22}$$

denotes the unit vector field tangent to the crack.

Let us suppose there exists a time interval $[\tau_0, \tau_1] \subset [0, \bar{\tau}]$ such that we get $l(\tau_0) > 0$ and $\mathrm{d}l/\mathrm{d}\tau > 0$ for $\tau \in [\tau_0, \tau_1]$; l restricted to this interval is an increasing function of the time and can therefore be used as a time scale. For $l \in [l_0, l_1]$ with $l_0 = l(\tau_0)$, $l_1 = l(\tau_1)$, the crack tip advances without stopping in direction $e(l)$.

For each $l \in [l_0, l_1]$ and for each small $\delta > 0$, $\mathscr{D}_\delta(l)$ denotes the disc of radius δ centred at the crack tip and $\mathscr{B}_\delta(l) = \mathscr{B} - \mathscr{D}_\delta(l)$ is the complement of $\mathscr{D}_\delta(l)$ with respect to $\mathscr{B}$. v, and n are the outward unit normal to $\partial\mathscr{B}$ and $\partial\mathscr{D}_\delta$, respectively (Fig. 1). For $x \in \mathscr{B}$ and $l \in [l_0, l_1]$, we use $u(x, l)$ to indicate the displacement field with respect to the reference configuration and put

$$E(x, l) = \hat{E}_x(\tau), \qquad E^{\mathrm{p}}(x, l) = \hat{E}_x^{\mathrm{p}}(\tau) \tag{23}$$

$$T(x, l) = \mathbb{T}[E(x, l) - E^{\mathrm{p}}(x, l)] \tag{24}$$

$$\zeta(x, l) = \zeta_{\mathrm{E}_x}(\tau) \tag{25}$$

$$w(x, l) = \hat{w}_{\mathrm{E}_x}(\tau) \tag{26}$$

where ζ and $\hat{w}$ are the functions defined by (13) and (18), respectively, and where $l = l(\tau)$.

Let $\phi(x, l)$, with $x \in \mathscr{B}$ and $l \in [l_0, l_1]$, be a smooth field. We put $\phi' = \partial\phi/\partial l$. Moreover, if $\phi(x, l)$ is a scalar or vector field, $\nabla\phi$ denotes the gradient of ϕ with respect to x; if ϕ is a vector or tensor field, div ϕ indicates the divergence of ϕ.

Let us now go on to list certain properties. At this point it may be worthwhile noting that properties **I3** and **I4** below, which are simply postulated here, could be deduced by means of the same procedures as those followed in **(3)**, from hypotheses of regularity on fields u, w, and T similar to those put forward in that paper.

Let us take $x \in \mathscr{B}$ and $l \in [l_0, l_1]$.

I1 Fields $u(x, l)$ and $E^p(x, l)$, $\zeta(x, l)$ are of class C^2 and C^1 on set $\mathscr{B} - \mathscr{C}(l)$, respectively. Moreover, $u(x, l)$, $E^p(x, l)$, $\zeta(x, l)$, and their derivatives are continuous up to the crack from either side, except at the tip.

I2 Function $w(x, l)$ defined by relation (26) and its derivative with respect to l are integrable on $\mathscr{B}$.

I3 $\int_{\mathscr{B}} w(x, l)\, da$ is differentiable with respect to l and

$$\lim_{\delta \to 0} \frac{d}{dl} \int_{\mathscr{B}_\delta} w(x, l)\, da = \frac{d}{dl} \int_{\mathscr{B}} w(x, l)\, da; \tag{27}$$

I4 We have

$$\lim_{\delta \to 0} \int_{\partial \mathscr{D}_\delta} Tn \cdot u'\, ds = -\lim_{\delta \to 0} e \cdot \int_{\partial \mathscr{D}_\delta} \nabla u^T T n\, ds \tag{28}$$

I5 Function $\operatorname{div}(wI - \nabla u^T T)$ is integrable on $\mathscr{B}$.

I6 $\mathscr{B}$ undergoes quasi-static deformations alone, in the absence of body forces and therefore

$$\operatorname{div} T(x, l) = 0 \tag{29}$$

I7 The crack faces are traction-free.

Remark

In view of **I1**, $E = \frac{1}{2}(\nabla u + \nabla u^T)$ and E^p are of class C^1 on $\mathscr{B} - \mathscr{C}(l)$ and, therefore, from (12) and (24) it follows that T is also of the same class. In the same way, w is of class C^1 on $\mathscr{B} - \mathscr{C}(l)$, in view of (19), (25) and **I1**.

The crack extension energy rate

If, in a certain interval of time, there is no propagation of the crack, in view of (18) and (26) the theorem of power expended **(11)** tells us that

$$-\frac{d}{d\tau} \int_{\mathscr{B}} w\, da + \int_{\partial \mathscr{B}} T\nu \cdot \dot{u}\, ds = 0 \tag{30}$$

In the interval $[l_0, l_1]$, where the crack advances, it is necessary to include crack extension energy rate $G(l)$ in the balance equation. In the case of hyperelastic bodies, $G(l)$ is called the *energy release rate*, and is defined by equation (1). In the case of elastic–plastic materials, in view of (30), as a generalization of (1) we put

$$G(l) = -\frac{d}{dl} \int_{\mathscr{B}} w\, da + \int_{\partial \mathscr{B}} T\nu \cdot u'\, ds \tag{31}$$

For each $l \in [l_0, l_1]$

$$\int_{l_0}^{l} G(\xi)\, d\xi$$

is interpreted here as the work per unit thickness needed to increase the length of the crack from l_0 to l.

The following proposition supplies an expression for $G(l)$ which is formally similar to that to be found in the case of hyperelastic bodies (see **(3)** equation (4.6)).

Proposition 4.1

For each $l \in [l_0, l_1]$

$$G(l) = \lim_{\delta \to 0} e(l) \cdot \int_{\partial \mathscr{D}_\delta} (wn - \nabla u^T T\mathbf{n})\, ds \tag{32}$$

Proof

From (18) and (29) we deduce

$$w' = T \cdot E' = T \cdot \nabla u' = \operatorname{div}(Tu') - u' \cdot \operatorname{div} T = \operatorname{div}(Tu')$$

Applying the divergence theorem to set $\mathscr{B}_\delta$, and bearing **I7** in mind, we have

$$\int_{\mathscr{B}_\delta} w'\, da = \int_{\partial \mathscr{B}} T\nu \cdot u'\, ds - \int_{\partial \mathscr{D}_\delta} Tn \cdot u'\, ds \tag{33}$$

The transport theorem **(11)** states that, for $\delta > 0$,

$$\frac{d}{dl} \int_{\mathscr{B}_\delta} w\, da = \int_{\mathscr{B}_\delta} w'\, da - e \cdot \int_{\partial \mathscr{D}_\delta} wn\, ds \tag{34}$$

and therefore from (33) it can be deduced that

$$-\frac{d}{dl} \int_{\mathscr{B}_\delta} w\, da + \int_{\partial \mathscr{B}} T\nu \cdot u'\, ds = \int_{\partial \mathscr{D}_\delta} (e \cdot wn + Tn \cdot u')\, ds \tag{35}$$

The desired conclusion now follows from **I3**, **I4** and (31).

A curve γ is called a *path around the tip* if it is a smooth non-intersecting path that starts and ends on the crack and includes the tip of the crack (Fig. 2). Let γ be a path around the tip and n the outward unit normal on γ. The quantity

$$J(\gamma) = e \cdot \int_{\gamma} (wn - \nabla u^T T n)\, ds \tag{36}$$

is called the *J-integral* for the path γ.

In the same way as in the case of hyperelastic materials (see **(3)** equation (5.2)), we have the following, in view of (32) and (36),

$$G(l) = \lim_{\delta \to 0} J(\partial \mathscr{D}_\delta) \tag{37}$$

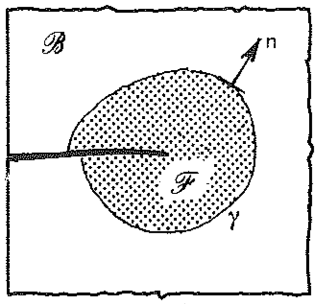

Fig 2 Path around the tip

For each $l \in [l_0, l_1]$ let

$$\mathscr{B}_\zeta(l) = \{\boldsymbol{x} \in \mathscr{B} / \zeta(\boldsymbol{x}, l) \neq 0\}, \tag{38}$$

with $\zeta(\boldsymbol{x}, l)$ defined by (25), be the set of points of $\mathscr{B}$ in which plastic deformations have taken place. Let us call $\mathscr{B}_\zeta(l)$ the plastic region; moreover, it should be noted that $\mathscr{B}_\zeta(l)$, in general, does not coincide with the subset of $\mathscr{B}$ in which we have $E^p(\boldsymbol{x}, l) \neq 0$.

For each path γ around the tip, let $\mathscr{F}$ be the subset of $\mathscr{B}$ enclosed by γ and let $\mathscr{L} = \mathscr{F} \cap \mathscr{B}_\zeta$ be the intersection of $\mathscr{F}$ and $\mathscr{B}_\zeta$ (Fig. 3).

Proposition 4.2

If the crack is straight, for each path γ around the tip we have

$$G(l) = J(\gamma) - \boldsymbol{e} \cdot \int_{\mathscr{L}} \operatorname{div}\,(wI - \nabla \boldsymbol{u}^{\mathrm{T}} T)\,\mathrm{d}a \tag{39}$$

Proof

Let $\delta > 0$ be small enough for path γ to include $\mathscr{D}_\delta$ and let $\mathscr{F}_\delta$ be the subset of $\mathscr{B}$ bounded by γ, $\partial\mathscr{D}_\delta$ and the two faces of the crack. Taking **I7** into account and bearing in mind that the crack is straight, the divergence theorem applied

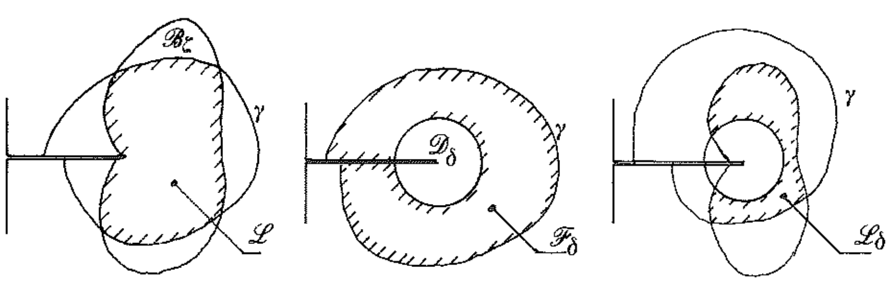

Fig 3 Plastic region

to region $\mathscr{F}_\delta$ tells us that

$$J(\partial\mathscr{D}_\delta) = J(\gamma) - \boldsymbol{e} \cdot \int_{\mathscr{F}_\delta} \operatorname{div}\,(wI - \nabla \boldsymbol{u}^{\mathrm{T}} T)\,\mathrm{d}a \tag{40}$$

On the other hand, at all the points $\boldsymbol{x}$ not belonging to $\mathscr{B}_\zeta$, $w(\boldsymbol{x}, l)$ coincides with the strain–energy density and therefore, as proved in **(4)**, at such points we have $\operatorname{div}\,(wI - \nabla \boldsymbol{u}^{\mathrm{T}} T) = 0$. Given, then, that

$$\mathscr{L}_\delta = \mathscr{F}_\delta \cap \mathscr{B}_\zeta$$

we obtain the following from (40)

$$J(\partial\mathscr{D}_\delta) = J(\gamma) - \boldsymbol{e} \cdot \int_{\mathscr{L}_\delta} \operatorname{div}\,(wI - \nabla \boldsymbol{u}^{\mathrm{T}} T)\,\mathrm{d}a \tag{41}$$

The desired conclusion is obtained by taking the limit for $\delta \to 0$ in equation (41), bearing (37) and **I5** in mind.

If, for some $l \in [l_0, l_1]$, the distance between $\mathscr{B}_\zeta(l)$ and $\partial\mathscr{B}$ is positive, there exist paths around the tip that include $\mathscr{B}_\zeta(l)$. If γ is one of these particular paths around the tip, $\mathscr{F}$ contains $\mathscr{B}_\zeta$ and it therefore follows from (39) that

$$G(l) = J(\gamma) - \boldsymbol{e} \cdot \int_{\mathscr{B}_\zeta(l)} \operatorname{div}\,(wI - \nabla \boldsymbol{u}^{\mathrm{T}} T)\,\mathrm{d}a \tag{42}$$

Moreover, since γ does not intersect with $\mathscr{B}_\zeta$, in view of (19) $J(\gamma)$ can be calculated, rather than from (36), from (2) as in the case of hyperelastic materials.

If for some $\boldsymbol{x} \in \mathscr{B}$ and $l \in [l_0, l_1]$ we have $\zeta(\boldsymbol{x}, l) = 0$, in view of (13) and (19) $w(\boldsymbol{x}, l)$ coincides with the strain–energy density and so **(4)**

$$\operatorname{div}\,(w(\boldsymbol{x}, l)I - \nabla \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{x}, l)T(\boldsymbol{x}, l)) = 0 \tag{43}$$

With the following proposition it is proved that, even for $\boldsymbol{x} \in \mathscr{B}_\zeta$ there exists a particular circumstance in which equation (43) holds.

Proposition 4.3

If for $\boldsymbol{x} \in \mathscr{B}_\zeta(l)$

$$\|E^p(\boldsymbol{x}, l)\| = \zeta(\boldsymbol{x}, l) \quad \text{and} \quad E_0(\boldsymbol{x}, l) = C(\boldsymbol{x}, l) + (\rho(\zeta)/\zeta)E^p(\boldsymbol{x}, l) \tag{44}$$

then equation (43) holds.

As can be deduced from the constitutive equations (13)–(17), condition (44) is satisfied if, in particular, at point $\boldsymbol{x}$ the deformation process $\hat{E}_x$ is proportional and monotonous. That is to say, if there exists a symmetrical tensor E_x^0 with $\|E_x^0\| = 1$, such that

$$\hat{E}_x(\tau') = \|\hat{E}_x(\tau')\| E_x^0 \quad \text{for each } \tau' \in [0, \bar{\tau}]$$

where $\|\hat{E}_x(\tau')\|$ is a non-decreasing function of τ'.

Proof

Let us put

$$\psi(\tau) = \tfrac{1}{2}(\hat{E}(\tau) - \hat{E}^{p}(\tau)) \cdot \mathbb{T}[\hat{E}(\tau) - \hat{E}^{p}(\tau)] + \mu\eta\|\hat{E}^{p}(\tau)\|^2$$

from (19) we deduce

$$\mathrm{div}\,(wI) = \nabla(\psi + 2\mu\omega(\zeta)) = \nabla\psi + 2\mu\rho(\zeta)\nabla\zeta \qquad (45)$$

If $\boldsymbol{a}$ is any vector of $\mathbb{R}^2$, in view of (20) we have†

$$\begin{aligned}\boldsymbol{a}\cdot\nabla\psi &= 2\mu(E - E^{p})\cdot\partial(E - E^{p})[\boldsymbol{a}] + 2\mu\eta E^{p}\cdot\partial E^{p}[\boldsymbol{a}] \\ &\quad + (\lambda\,\mathrm{tr}\,E)I\cdot\partial(E - E^{p})[\boldsymbol{a}] \\ &= T\cdot\partial(E - E^{p})[\boldsymbol{a}] + 2\mu\eta E^{p}\cdot\partial E^{p}[\boldsymbol{a}] \end{aligned} \qquad (46)$$

Moreover, from (44) it follows that

$$2\mu\rho(\zeta)(\nabla\zeta\cdot\boldsymbol{a}) = 2\mu\rho(\zeta)(E^{p}/\|E^{p}\|)\cdot\partial E^{p}[\boldsymbol{a}] \qquad (47)$$

Equations (45), (46), (47), (44), (12) and the fact that E^p is a traceless tensor imply

$$\begin{aligned}\boldsymbol{a}\cdot\mathrm{div}\,(wI) &= T\cdot\partial(E - E^{p})[\boldsymbol{a}] + 2\mu(\eta + \rho(\zeta)/\zeta)E^{p}\cdot\partial E^{p}[\boldsymbol{a}] \\ &= T\cdot\partial(E - E^{p})[\boldsymbol{a}] + T_0\cdot\partial E^{p}[\boldsymbol{a}] \\ &= T\cdot\partial(E - E^{p})[\boldsymbol{a}] + T\cdot\partial E^{p}[\boldsymbol{a}] = T\cdot\partial E[\boldsymbol{a}] \end{aligned} \qquad (48)$$

On the other hand, bearing (29) and the symmetry of T in mind, we have

$$\begin{aligned}\boldsymbol{a}\cdot\mathrm{div}\,(\nabla\boldsymbol{u}^{\mathrm{T}}T) &= \mathrm{div}\,(T\nabla\boldsymbol{u}\boldsymbol{a}) = T\cdot\nabla(\nabla\boldsymbol{u}\boldsymbol{a}) \\ &= T\cdot(\partial\nabla\boldsymbol{u}[\boldsymbol{a}]) = T\cdot\partial E[\boldsymbol{a}] \end{aligned} \qquad (49)$$

The desired conclusion is now a direct consequence of (48) and (49).

Propositions 4.2 and 4.3 imply the following corollary.

Corollary 4.1

If the crack is straight and for each $\boldsymbol{x} \in \mathscr{B}_\zeta(l)$ condition (44) is verified we have

$$G(l) = J(\gamma) \qquad (50)$$

for each path γ around the tip.

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† If $A: \mathscr{B} \to \mathrm{Sym}$ is a tensor field on $\mathscr{B}$, $\boldsymbol{x}$ is a point of $\mathscr{B}$ and $\boldsymbol{a} \in \mathbb{R}^2$ is a vector, we put

$$\partial A(\boldsymbol{x})[\boldsymbol{a}] = \lim_{\varepsilon\to 0} \varepsilon^{-1}(A(\boldsymbol{x} + \varepsilon\boldsymbol{a}) - A(\boldsymbol{x})).$$

References

(1) IRWIN, G. R. (1958) Fracture, in *Handbuch der Physik, Vol. VI, Elasticity and Plasticity*, pp. 551–590, (Edited by S. Flugge), Springer-Verlag.

(2) RICE, J. R. (1968) A path independent integral and the approximate analysis of strain concentrations by notches and cracks, *J. Appl. Mech.*, **35**, 379–386.

(3) GURTIN, M. (1979) On the energy release rate in quasi-static elastic crack propagation, *J. Elasticity*, **9**, 187–195.

(4) ESHELBY, J. D. (1970) Energy relations and the energy-momentum tensor in continuum mechanics, *Inelastic Behavior of Solids*, 75–115, (Edited by M. F. Kanninen *et al*), McGraw-Hill, New York.

(5) SCHMITT, W. and KIENZLER, R. (1989) The *J*-Integral concept for elastic–plastic material behavior, *Engng Fracture Mech.*, **32**, 409–418.

(6) MORAN, B. and SHIH, C. H. (1987) A general treatment of crack tip contour integrals, *Int. J. Fracture*, **35**, 295–310.

(7) BENNATI, S. (1980) Un estensione dell'Integrale J tridimensionale, *Atti dell'Istituto di Scienza delle Costruzioni dell'Università di Pisa*, **17**, 3–23.

(8) LUCCHESI, M., OWEN, D. R., and PODIO GUIDUGLI, P. Materials with elastic range: A theory with a view toward applications. Part III. Arch. Rational Mech. Anal., in press.

(9) MELAN, E. (1938) Zur Plastizität des raumlichen Kontinuums. *Ing. Arch.*, **9**, 116–126.

(10) LUCCHESI, M. Free energy functions for elastic–plastic material element. Submitted for publication.

(11) GURTIN, M. (1981) An introduction to continuum mechanics. Academic Press.

(12) RICE, J. R. (1979) The mechanics of quasi-static crack growth. *Proc. 8th U.S. National Congress of Applied Mechanics*, (Edited by R. E. Kelly), pp. 191–216, Western Periodicals.

(13) MORAN, B. and SHIH, C. H. (1987) Crack tip and associated domain integrals from momentum and energy balance, *Engng Fracture Mech.*, **27**, 615–642.