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Crack Growth Onset in Biaxially Loaded Elasto-Plastic Plates

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ABSTRACT The results of an experimental investigation concerning the occurrence of spontaneous slip bands and crack growth initiation in biaxially loaded thin mild-steel plates are analysed. The so-called ρ theory of fracture and the plane stress crack tip blunting model are used to explain the observed biaxial stress effects on the characteristic values of stress intensity factor K_1, the J_1 integral, and the crack tip opening displacement δ_1. It is shown that the influence of the load biaxiality on characteristic values of fracture parameters not only increases but also undergoes certain qualitative changes during the transition from brittle fracture to elasto-plastic and then to fully plastic fracture.

Notation

P, Q	Nominal loads remote from the test area acting normal to and along the crack line
σ, q	Remote nominal stresses arising from loads P and Q
$\sigma_1, \sigma_2, \sigma_3$	Principal stresses within the test area ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$)
k	Nominal biaxial stress ratio (q/σ)
λ	Local biaxial stress ratio (σ_2/σ_1 or σ_3/σ_1)
K_1	Stress intensity factor
J_1	The J-integral
δ_1	Crack tip opening displacement
v	Displacement due to load P measured in points remote from the crack line
W	Half-width of the specimen
a_0	Half-length of the original crack
ρ, a	The minimum curvature radius and the major axis half-length of the elliptical hole equivalent to the real crack in the ρ-theory
l	Half-length of the slit representing a real crack in the ρ-theory
p	Averaged cohesion forces near the crack tip
U, U_d, U_p	The full work of fracture expended on specimen tension and its components in the process zone and in the external plasticity zone
d	The process zone dimension
ω	The stress parameter in the slip-line theory
$\Omega(\omega)$	The function of ω

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θ	The first principal angle
(r, γ)	Polar coordinates
σ_y	Yield strength
σ_u	Ultimate strength

Subscripts

h	Any characteristic moment of the fracture process
e	The moment of plasticity bands occurrence
i	The crack growth onset
c	The unstable crack expansion onset

Introduction

The limiting state of the opening mode through cracks in thin-walled parts of shell-type structures is usually determined from the data obtained on uniaxially loaded specimens. This approach is based on the assumption that the fracture toughness characteristics are independent of the load acting along the crack line. However, the results of numerous investigations prove the converse. Thus the literature mainly contains theoretical, predominantly numerical, papers indicating the dependence of the plastic zone dimensions, the stresses and strains near the crack tip, and values of the fracture parameters K_1, J_1, δ_1 on the biaxial stress ratio k, which is equal to the ratio of nominal stresses q and σ shown in Fig. 1. A smaller group of experimental papers usually contain information concerning the influence of the ratio k on the critical load value σ and rarely concerning the effect of load biaxiality on the characteristic values of fracture parameters.

Shortage and certain contradictions in the published data do not permit unambiguous conclusions to be drawn about the nature and degree of the dependence of the characteristic values of the fracture parameters on load biaxiality. This paper presents and re-interprets the results of experimental studies of the opening mode fracture in sheet mild steel with biaxial loading **(1)** (The experiments were carried out by E. E. Onishchenko with the participation of the authors of this paper.) Our attention is focused on the moment of spontaneous occurrence of the first slip bands near the crack tip and the stable crack growth onset. An analysis of the experimental results is carried out on the basis of original models for predicting the limiting state of mode 1 crack under biaxial loading in two extreme cases of elasto-plastic fracture: linearly elastic and fully plastic fracture. A generalised representation of these effects for intermediate cases is also proposed.

Experimental procedure and results

The material tested was a 4 mm thick, rolled sheet of low-carbon mild steel with the yield strength $\sigma_y = 220$ MPa and ultimate strength $\sigma_u = 330$ MPa. The 'stress–strain' curve of a smooth specimen under uniaxial tension and the

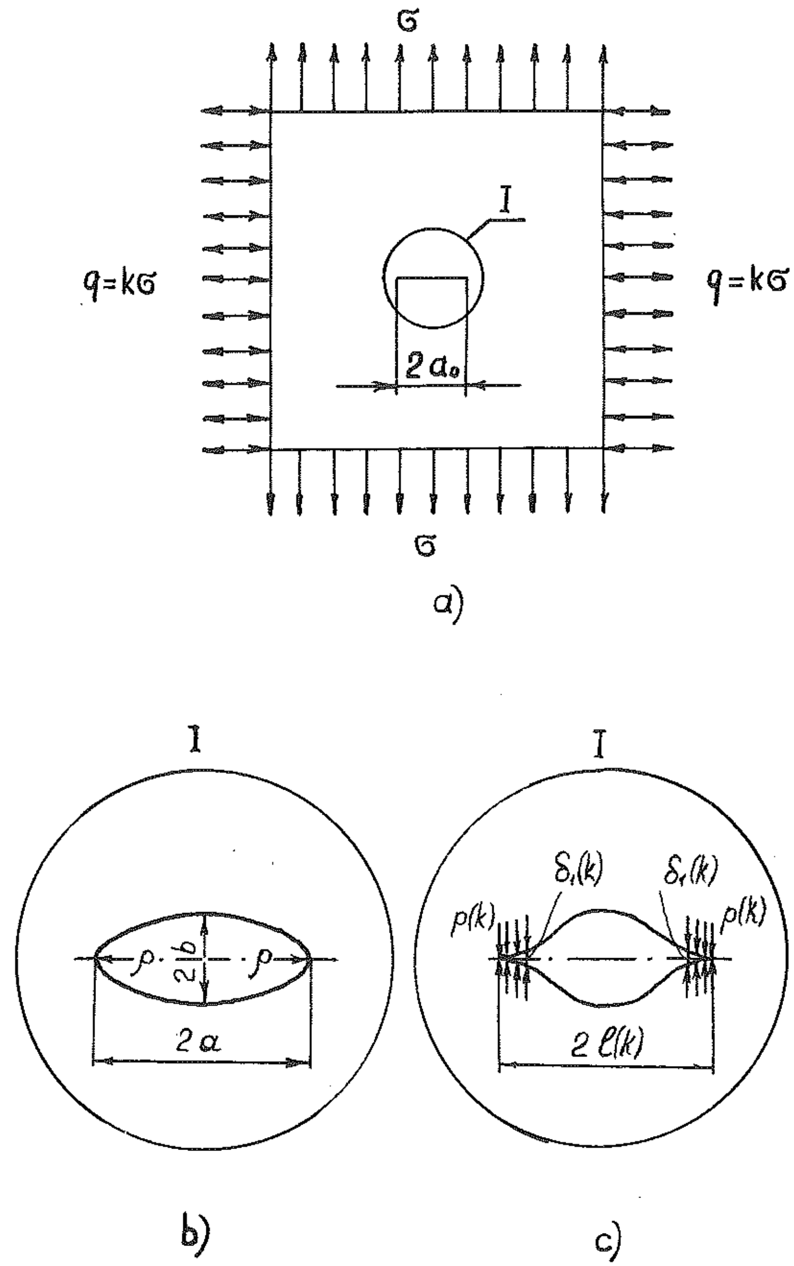

Fig 1 **Opening mode crack in (a) a biaxially loaded infinite plate (b) an elliptical hole and (c) a partly loaded slit which represent the real crack in the ρ theory of fracture and in the modified Barenblatt–Dugdale–Leonov–Panasyuk model, respectively**

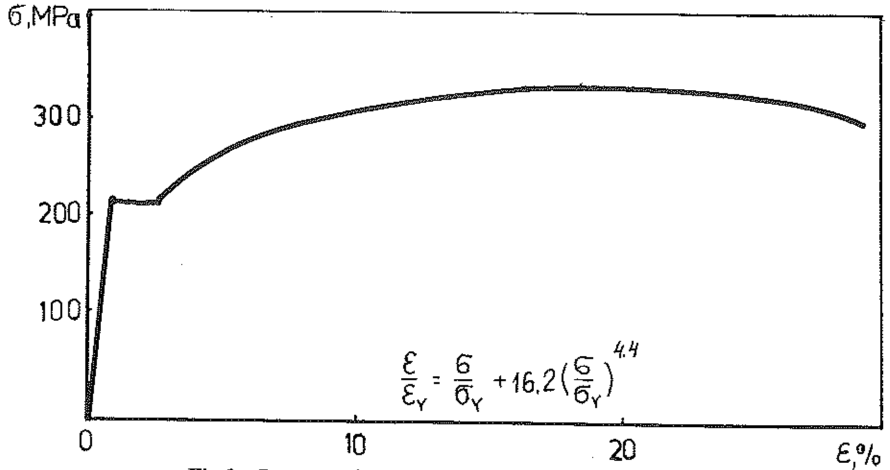

Fig 2 Stress–strain curve for tested low-carbon mild steel

strain hardening law for this steel are presented in Fig. 2. Plane M(T) and cruciform specimens with a central crack were tested (see Fig. 3). The specimens had fatigue cracks. The crack length including initial sharp notches was $2a_0 = (0.43 \ldots 0.53) \times 2 \times W$.

Cruciform specimens were tested in a special device **(2)**, ensuring independent loading along and across the crack line. At first specimens were loaded by force Q up to a preset stress level, which was then kept constant. They were then subjected to tension by force P to final fracture. The range of force Q variation was $(-150 \ldots 150)$ kN.

The J_1-integral was determined from the area under the load–displacement $(P - v)$ diagram using the formula for M(T) specimens developed by Rice *et al.* **(3)**. Here v is the displacement of specimen points indicated by arrows in Fig. 3. The value of δ_1 was measured between points on the intersection of the crack front line and the specimen side surface on the slides made during the test. In order to determine the moment of the crack growth initiation, visual examination of the blunting part of the crack surface was carried out with a microscope installed at an angle to the specimen plane. The crack growth onset was identified as an initiation which could be distinctly observed, quickly followed by the coalescence of the first two or three pores on the crack surface.

The form of the P–v diagram depends essentially on load Q level, at least when Q is compressive. There is no noticeable effect of Q on the slope of the P–v diagram linear portions, while the load P_e limiting the latter greatly depends on the ratio k. At $P = P_e$ on the side surface of specimens the spontaneous occurrence of a plasticity zone was observed in the form of narrow bands $10 \ldots 15$ mm in length, resembling the formation of Chernov–Lüders slip bands on polished surfaces of smooth specimens. In the $M(T)$ specimens the slip bands occurred in the crack plane and in the cruciform specimens one

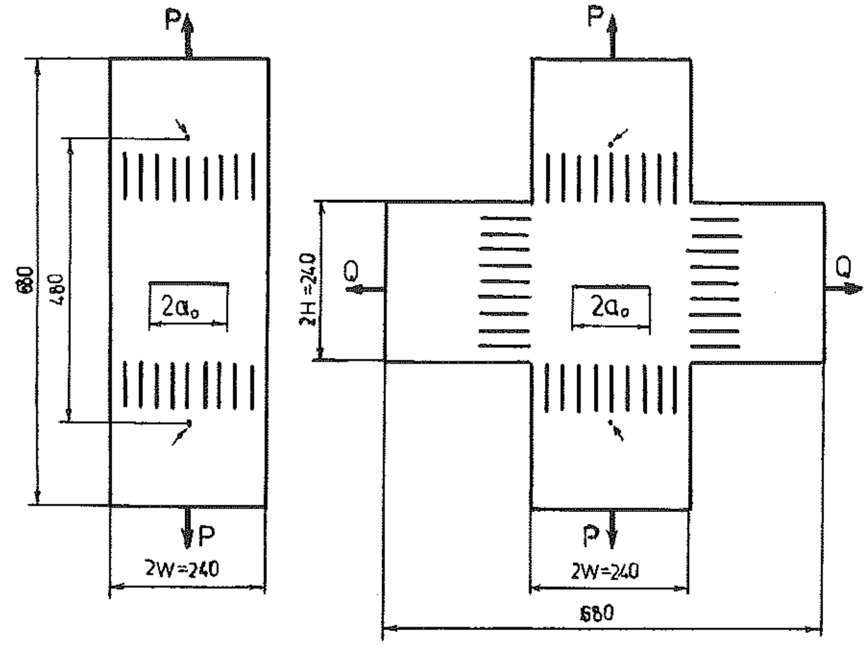

Fig 3 Centrally cracked M(T) and cruciform specimens. The arrows indicate measurement points of displacement v

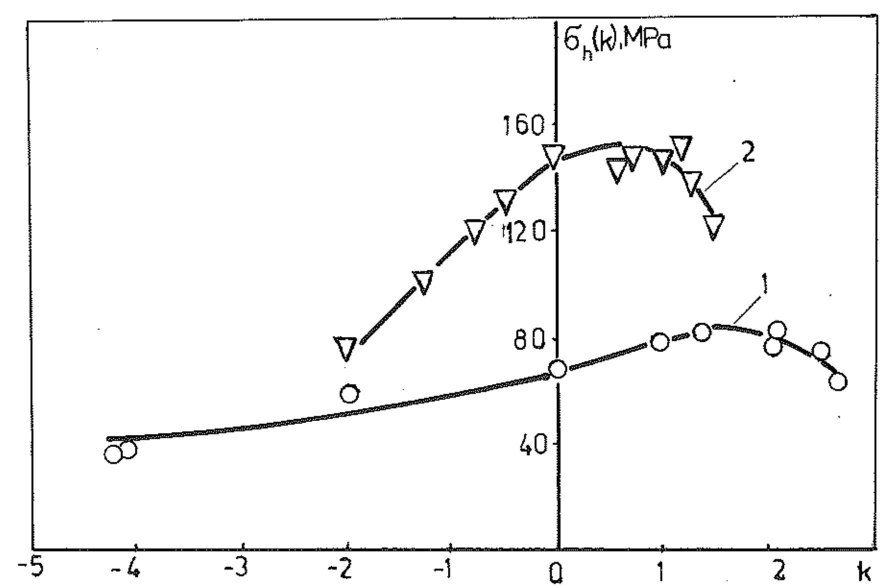

Fig 4 Effect of biaxial stress ratio k on the characteristic values of stress σ at the moment of slip band occurrence (1) and at the onset of crack growth (2). Here, and in Figs 5 and 6, the points correspond to experimental data; the lines are the best fit lines

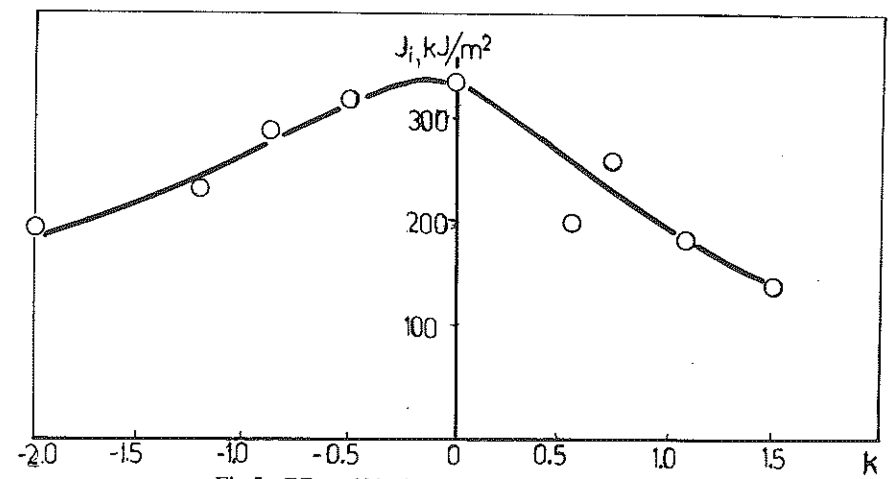

Fig 5 Effect of biaxial stress ratio k on the J_i values

horizontal and two inclined bands generally appeared simultaneously at any value of k. In all cases the crack growth onset was observed when the plasticity zone had stretched over the ligament of the specimens.

The experimental values of nominal stresses σ, corresponding to the moments 'e' (occurrence of plasticity bands) and 'i' (the crack growth onset), are shown in Fig. 4. The dependence shown is dome-shaped for both limiting states. Similar dome-shaped curves of the load variation for crack growth onset were obtained in references **(4)** and **(5)** for non-metallic materials and aluminium alloys, respectively. The dependence of the crack growth onset values J_i and δ_i on the biaxial stress ratio shown in Figs 5 and 6, has the same shape.

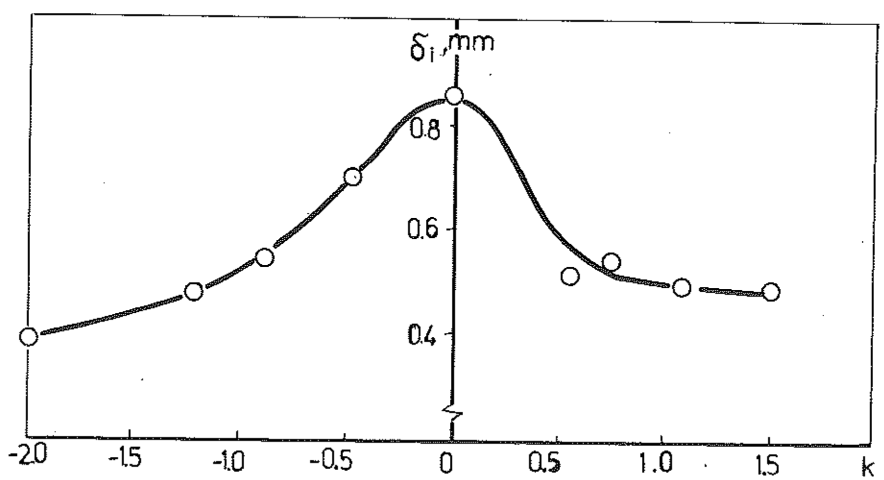

Fig 6 Effect of biaxial stress ratio k on the δ_i values

Analysis

The spontaneous occurrence of slip bands The diagram linearity up to the moment 'e' and the relatively small size of slip bands permit the use of the so-called ρ-theory of mode 1 fracture **(4)**. According to the theory for linearly elastic biaxially loaded infinite plates with a central crack (Fig. 1(a)) we shall have

$$K_1(k) = \sigma\sqrt{\{\pi l(k)\}} \tag{1}$$

where

$$l(k) = a + 0.5\rho k^2 + 0.375\sqrt{(\rho a)}(1 - k)^2 \tag{2}$$

is the half-length of the slit representing a real crack with non-zero tip curvature radius.

In formula (2), a is the semi-major axis length of the 'equivalent' elliptical hole (Fig. 1(b)) equal to a half-length of the crack a_0; ρ is the minimum curvature radius of the hole ($\rho = b^2/a$).

The material characteristic ρ is determined according to the results of two tests carried out with different values of the ratio k. The optimal tests for brittle materials can be carried out at $q = 0$ and at $\sigma = 0$. When determining the value of ρ for the ductile steel tested the data obtained under uniaxial tension and at $k = -4.2$ were used.

Assuming that the characteristic K_e, equal to the K_1 value from (1) at the moment 'e', is constant in the investigated range of the k ratio variation, in the case of fixed crack length we shall obtain

$$\frac{\sigma_e(k)}{\sigma_e(0)} = \left(\frac{a + 0.375\sqrt{(\rho a)}}{a + 0.5\rho k^2 + 0.375\sqrt{(\rho a)}(1 - k)^2}\right)^{1/2} \tag{3}$$

For the steel tested we obtain $\rho = 1.3$ mm from formula (3) at

$$a = a_0 = 55 \text{ mm} \qquad k = -4.2 \quad \text{and} \quad \sigma_e(k)/\sigma_e(0) = 0.62$$

Using this value of ρ it is possible to calculate the dependence of the stress σ_e on the ratio k for the tested plates by formula (3). The predicted stress practically coincides with the experimental data in the region of negative k (Fig. 7). The qualitative correspondence and noticeable quantitative differences are observed under biaxial tension. The latter can be caused by non-adherence to or deviation from the assumptions used in the calculation and by variation of the a_0 dimension. At the same time the stress σ_e predicted for the cruciform specimen with two-fold decreased crack length $a_0 = 22.5$ mm under biaxial tension with $k = 1.56$ practically coincided with the experimentally measured value. In the calculations the value of $K_e = 36.6$ (MPa$\sqrt{m}$) obtained from the $M(T)$ specimen with a crack length $a_0 = 67$ mm was used. The value of K_e was calculated taking into account the finiteness of the specimen central square dimension according to paper **(6)**, and the dimension $l(k)$ from formula

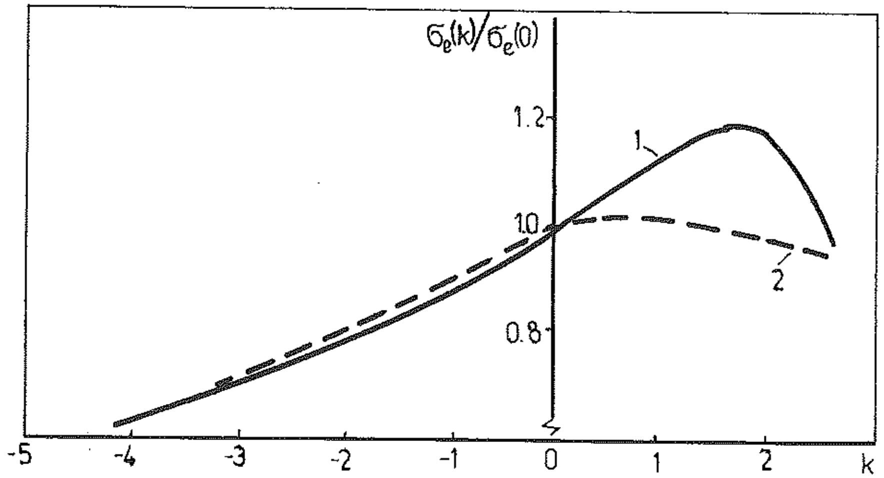

Fig 7 Comparison of stress σ_e (1) experimentally obtained and (2) calculated from formula (3)

(2) was substituted. On the basis of the results of reference **(2)** K_1-calibrations for the cruciform and $M(T)$ specimens were assumed to be slightly different. In any case the discrepancy between K_e values calculated in the above-mentioned way for the $M(T)$ spcimen and those for the uniaxial tensile cruciform specimen does not exceed 10 percent.

The results obtained show that the value K_e is independent of or only very slightly dependent on the load biaxiality.

The crack growth onset The values of J_i and δ_i at the crack growth onset (Figs 5, 6) appeared to be highly sensitive to the k ratio, i.e., variable with the type of plate loading. As the stable crack growth occurred under full-scale yielding conditions, in order to explain the experimental results it is advisable to examine the fracture of a thin rigid-plastic non-hardening plate with a central crack, assuming the von Mises yielding condition

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2 \quad (4)$$

where σ_1 and σ_2 (or σ_3, if the minimum stress is negative) are principal stresses in the plasticity zone.

The region of plastic strains in such a plate may be considered as a triangle limited by characteristic lines originating from the crack tip and the lateral edge of the specimen. The angle between these lines depends on the stress q, acting on the specimen edge. It is equal to **(7)**

$$2\phi = \cos^{-1}\left(\frac{1}{3}\frac{1+\lambda}{1-\lambda}\right) \quad (5)$$

where λ is the ratio of the principal stresses acting in the plasticity zone along and across the crack line. Assuming uniform distribution of stresses in the

plasticity zone (as characteristic lines are straight) we shall obtain $\sigma_2 = q$ and σ_1 is calculated from (4).

The work U expended on the tensile loading of the specimen to fracture can be approximately expressed as a sum of the work U_d and U_p expended in the process zone and in the external plasticity zone. By analogy with paper **(8)** it is possible to show that at the lowest λ values (less than 0.4) the work U_d is much less than work U_p, and for specimens with the same $W - a_0$ dimension

$$\frac{U(\lambda)}{U(0)} \approx \frac{\sigma_1(\lambda)}{\sigma_y}\frac{tg\phi(\lambda)}{tg\phi(0)} \quad (6)$$

Since the J_1-integral is directly proportional to U at the full-scale yielding regime, taking into account (4) we obtain

$$\frac{J_i(\lambda)}{J_i(0)} \approx \frac{1}{(1-\lambda+\lambda^2)^{1/2}}\frac{tg\phi(\lambda)}{tg\phi(0)} \quad (7)$$

The calculation carried out according to formula (7) agrees with the experimental data (Fig. 8). Hence the values of J_i obtained under full-scale yielding characterise the dissipation of strain energy in the external plasticity zone rather than the crack initiation resistance of the material in the process zone.

Unlike the J_1-integral the crack opening displacement δ_1 is connected directly with the process zone and shows the level of strains in this zone. Its dependence on the biaxial stress ratio k can be explained by an investigation of the crack tip blunting under plane stress conditions similar to the analysis

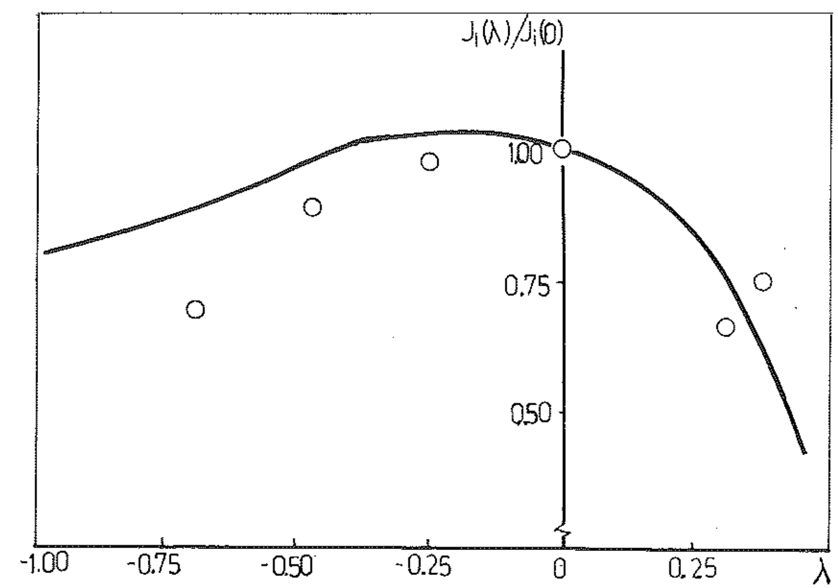

Fig 8 Experimentally determined (points) and predicted (line) values of J_i versus stress ratio λ

made for plane strain by Rice and Johnson (9). Our analysis is based on the following assumptions, which are consistent with the experimental observations: (1) crack growth begins at the moment when the process zone size d (Fig. 9) reaches the value d_i characteristic of the materials tested. The latter can be interpreted as a microstructural parameter, for example the distance between intermetallic inclusions; (2) the contour of the blunted crack tip is approximated by an arc of a circle limited by the intersection points of the fatigue crack front and the specimen side surface (Fig. 9). In this case the relationship between the crack opening displacement δ_1 and the size d can be established by solving the problem of plane stress yielding near the rounded notch root in non-hardening plate of von Mises material (7).

This relation has the following form (details of the derivation are given in the Appendix)

$$\frac{d}{\delta_1} = \frac{\{(\sqrt{3}/2)(1/\sin\omega_D)\}^{1/2}\exp[(\sqrt{3}/2)\{(\pi/3)-\omega_D\}]-1}{2\sin|(tg^{-1}[\{\sin(\omega_0+\pi/6)\}/\{\sin(\omega_0-\pi/6)\}]^{3/2}-1.2309)|} \tag{8}$$

Here, when $0 \leqslant \lambda \leqslant 0.5$

$$\omega_D = \omega_B = ctg^{-1}\left(\frac{1}{\sqrt{3}}\frac{1+\lambda}{1-\lambda}\right) \tag{9}$$

and when $-0.65 \leqslant \lambda < 0$ the stress parameter ω_D can be found from the equation

$$tg^{-1}\left[-\frac{\sin(\omega_0+\pi/6)}{\sin(\omega_D-\pi/6)}\right]^{3/2} + tg^{-1}\left[\frac{\sin(\omega_B+\pi/6)}{\sin(\omega_B-\pi/6)}\right]^{3/2} = 2.4619 \tag{10}$$

where ω_B is given by (9).

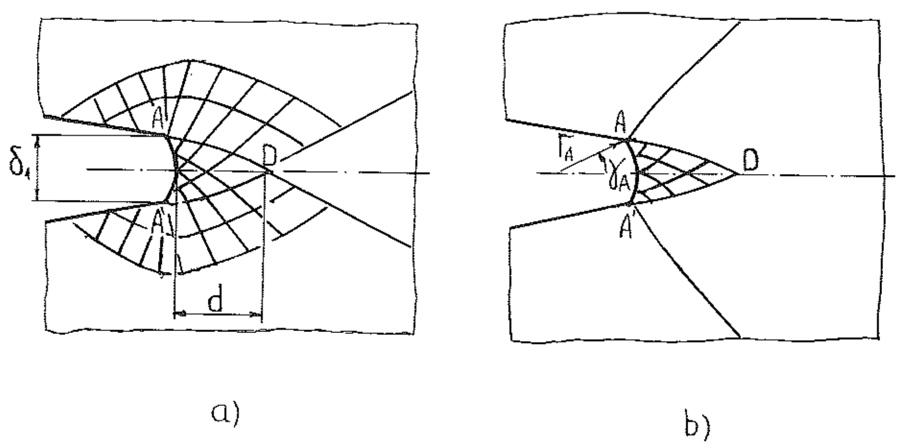

Fig 9 Characteristic line field nearby blunted crack tip in biaxially loaded plate for the cases of positive (a) and negative (b) values of the local stress biaxiality ratio

In the case of uniaxial tension the numerator and denominator of formula (8) become zero, however

$$\lim_{\lambda \to 0} \frac{d}{\delta_1} = 0.71$$

This means that at uniaxial tension the contour of the blunted crack tip is an arc with an infinite radius, i.e. the straight line segment and the region ADA′ in Fig. 9 are converted into a triangle. This agrees with the results of reference **(11)**. Owing to the limits imposed on the parameter ω **(7)**, relationship (8) is true only in the range of λ varying from -0.65 to 0.5, which embraces those cases of elasto-plastic fracture which are most important from the practical point of view.

Expression (8) is obtained within the frameworks of a rather simplified model and ignores factors such as strain hardening, the size of voids and inclusions, local thinning of the plate, etc. However, the calculated ratio $\delta_i(\lambda)/\delta_i(0)$ is in satisfactory agreement with the experimental data (Fig. 10). This indicates the sensitivity of the crack tip opening displacement δ_i at the crack growth onset to the type of loading under full-scale yielding conditions.

Generalised representation of biaxial stress effects The linearly elastic and fully plastic analysis made for the two characteristic states of a crack shows that the biaxiality effects observed are associated with different causes. In the first case, the stress q affects essentially parameter K_1 and almost does not change characteristic K_e. In the second case, biaxial stress affects the relation between the δ_i value and the microstructure parameter d_i. If the latter is a

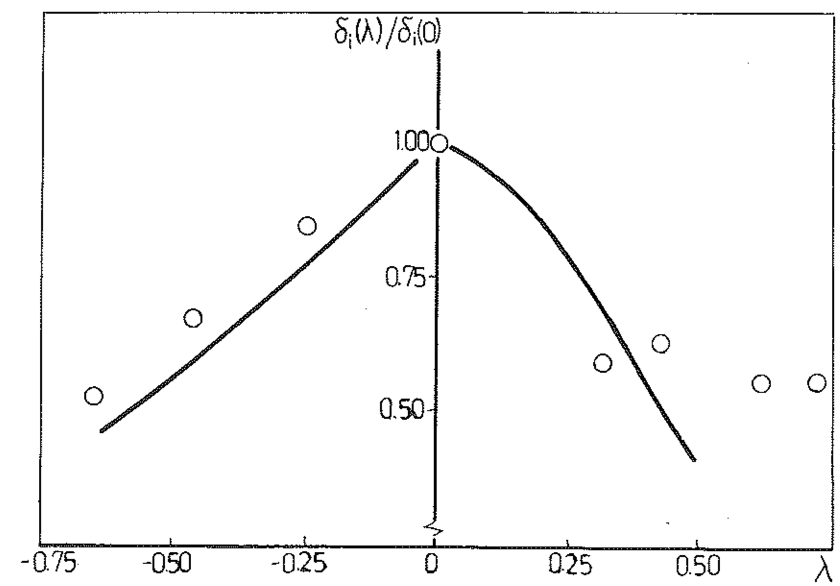

Fig 10 Experimentally determined (points) and predicted (line) values of δ_i versus stress ratio λ

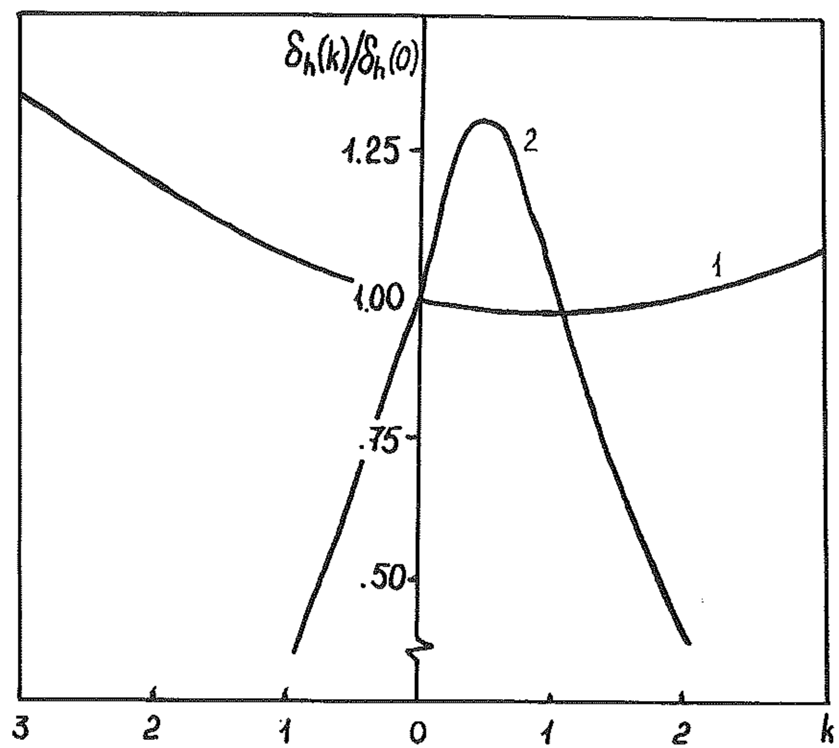

Fig 11 Generalised representation of the effects of load biaxiality on characteristic values of crack tip opening displacement δ_h for the cases of brittle fracture (1) and large-scale yielding (2)

material constant under the given test conditions, then δ_i must vary with the load biaxiality. Interrelated interpretation of these effects may be summarised for the intermediate case of elasto-plastic fracture by the Barenblatt-Dugdale-Leonov-Panasyuk model modified **(4)** for the case of biaxial loading (Fig. 1(c)). According to the model and assuming that $p(k) = p(0)$, where $p(k)$ are averaged cohesion forces, the relationship between the characteristic values of $\delta_1(k)$ has the following form

$$\frac{\delta_h(k)}{\delta_h(0)} \approx \left[\frac{\sigma_h(k)}{\sigma_h(0)}\right]^2 \left[\frac{l(k)}{l(0)}\right] \tag{11}$$

where subscript '*h*' means the correspondence of the δ_1 (k) values to any characteristic moment of the fracture process. The value of $l(k)$ is expressed by formula (2).

In the case of brittle fracture of material such as glass during classic experiments by Griffith reproduced in reference **(12)** it was established that at the onset of unstable crack extension in glass tubes loaded by inner pressure and axial compression the stress σ_c opening the crack depends to a slight extent on the k ratio ranging from -12 to 0. The δ_h–k relationship calculated according to formula (11) and the data from reference **(12)** for a glass plate with a crack ($a_0 = 55$ mm) is given in Fig. 11. The influence of load biaxiality results from $l(k)$ variation and is relatively small in the k ratio range from -1 to 1, which is typical of shell-type structures. However, the generally accepted assumption concerning the absence of the influence of load biaxiality on the crack growth resistance at brittle fracture can be considered to be true for a certain interval of k. It is confirmed by the phenomenon of the extension of isolated cracks under compression along the crack line at $\sigma = 0$, which is observed in excavations, structures of brittle non-metallic materials, and laboratory experiments **(4)**.

With a transition from the linearly elastic behaviour of material in fracture to elasto-plastic behaviour, i.e. with an increase in the size of the region of non-linear effects, the deviation of the first term in formula (11) from 1 increases and its influence on the shape of $\delta_h(k)$ dependence becomes predominant. If the material satisfies the von Mises yielding condition, the relationship σ_h-k will be determined according to formula (4) when the plasticity zone grows to dimensions comparable with the crack length. Therefore, for a larger region of plastic strain the dependence $\delta_h(k)$ assumes a dome-like shape (Fig. 11), with the maximum located between $k = 0$ and $k = 0.5$. Indeed, in reference **(5)** the maximum values of the parameters K_i and δ_i were obtained from cruciform specimens of aluminium alloys at $k \approx 0.5$.

Thus the effect of load biaxiality strongly depends on the size of the region of non-linear behaviour. An increase in the size leads not only to quantitative but also to qualitative changes in the dependence of parameters characterising a limiting state of biaxially loaded cracked plate on the biaxiality ratio k.

Conclusion

A new theoretical interpretation of a dome-shaped dependence of the crack opening load σ inducing the spontaneous occurrence of slip bands at the crack tip on the load $q = k\sigma$ applied along the crack line in thin-sheet mild steel is presented. It is in good agreement with the experiment.

The effects of load biaxiality on the J_1-integral and crack tip opening displacement δ_1 values at the onset of stable crack growth are explained. They are related to a change in the plasticity zone configuration with the varying ratio k in fully plastic biaxially loaded cracked plates.

Using a simple analytical model of elasto-plastic fracture mechanics a general structure of the opening mode fracture toughness dependence on load biaxiality is accounted for. It is also shown how and why the shape of such dependence is expected to change with the transition of the material from the brittle to the fully plastic state.

Appendix

In order to obtain relationship (8) assume the region of intensitive plastic strains near the blunted crack tip (the fracture process zone) to be bounded by characteristic lines AD and A'D (see Fig. 9). The configuration and dimension d of the zone depend on the external flow field. As a result of the low values of δ_1 and d compared with the ligament size, the external flow field can be considered identical to the flow field for a biaxially loaded plate with an acute slit. In the case of uniaxial tension this field is a triangle with the angle $2\phi = 70.3$ degrees (7). In the case of biaxial tension when $\lambda = 0.5$, the external flow field has the form proposed by Hutchinson **(10)**. For intermediate states when $0 \leqslant \lambda \leqslant 0.5$ such a field can be presented as a combination of the given fields (Fig. 9(a)).

Plane stress solution for the field of characteristics nearby the rounded notch root in the rigid ideally plastic von Mises plate is **(7)**

$$\left(\frac{r}{r_A}\right)^2 = \frac{\sqrt{3}}{2}\frac{1}{\sin\omega}\exp\left[\sqrt{3}\left(\frac{\pi}{3} - \omega\right)\right] \tag{12}$$

where r_A is the notch radius

$$\omega = ctg^{-1}\left(\frac{1}{\sqrt{3}}\frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2}\right) \tag{13}$$

is the stress parameter at an arbitrary point of the region ADA'; r is the distance from the centre of the notch root contour curvature.

As the crack tip opening displacement (Fig. 9)

$$\delta_1 = 2r_A \sin\gamma_A$$

therefore

$$\frac{d}{\delta_1} = \frac{1}{\sin\gamma_A}\left(\frac{\sqrt{3}}{2}\frac{1}{\sin\omega_D}\right)^{1/2}\exp\left[\frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - \omega_D\right)\right] \tag{14}$$

Along the characteristic lines the parameter ω and the first principal angle θ are related, i.e.

$$\theta \pm \Omega(\omega) = \text{const} \tag{15}$$

where function $\Omega(\omega)$ is (7)

$$\Omega(\omega) = tg^{-1}\left[\frac{\sin(\omega + \pi/6)}{\sin(\omega - \pi/6)}\right]^{3/2} \tag{16}$$

Since at the point A $\theta = \pi/2 + \gamma_A$, at the point D $\theta = \pi/2$, at the lateral bound of the plate (B) $\theta = \pi/2$, then

$$\frac{\pi}{2} + \gamma_A + \Omega\left(\frac{\pi}{3}\right) = \frac{\pi}{2} + \Omega(\omega_D) = \frac{\pi}{2} + \Omega(\omega_B) \tag{17}$$

Therefore

$$\omega_D = \omega_B = ctg^{-1}\left(\frac{1}{\sqrt{3}}\frac{1+\lambda}{1-\lambda}\right) \tag{18}$$

and

$$\gamma_A = \Omega(\omega_B) - \Omega\left(\frac{\pi}{3}\right) \tag{19}$$

Finally we obtain formula (8) from (14), (16), (18), (19).

In the case when $\lambda < 0$ we have to consider another scheme of the flow field nearby the rounded notch root (see Fig. 9(b)). The characteristic line reaching the border originates from the point A at the blunted crack tip. Along this line we have the negative sign in (15). The fracture process zone is bounded by the characteristic lines AD and A'D, and along AD we have the positive sign in (15). Then

$$\frac{\pi}{2} + \gamma_A - \Omega\left(\frac{\pi}{3}\right) = \frac{\pi}{2} - \Omega(\omega_B) \tag{20}$$

$$\frac{\pi}{2} + \gamma_A + \Omega\left(\frac{\pi}{3}\right) = \frac{\pi}{2} + \Omega(\omega_D) \tag{21}$$

As the result we obtain

$$\Omega(\omega_D) = 2\Omega\left(\frac{\pi}{3}\right) - \Omega(\omega_B) \tag{22}$$

and the relation between d and δ_1 in the same form as (8), but the parameter ω_D is to be found from equation (10).

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