# RELIABILITY ANALYSIS OF RC STRUCTURES. Towards an identification of importance zones in the random space.

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*The author compares several probabilistic procedures to analyze the reliability of an RC structure. A large part is devoted to the hypercone method, a geometrical technique, which affords upper-bound and lower-bound values for the probability of failure Pf. It is based on replacing the actual failure domain by sets of conical sections that contain the failure domain Df (case of the upper-bound value) or are contained in Df (case of lower-bound values).*

*This method tells also about the "probabilistic" importance of the various regions in the failure domain, giving then relevant informations about the sensitivity of the probability of failure.*

*The values of Pf obtained from the hypercone method are compared to the Monte Carlo simulations results. They are also compared to the operational values of Pf deduced from the Lind-Hasofer safety index.*

*The obtained values, in the case of an RC structure, show that Monte Carlo simulations and the safety index afford results that are very close to those of the hypercone method.*

*The hypercone method shows that restricted regions, in the operating random space, have major contributions in Pf values.*

## INTRODUCTION

The classical methods, that are used in reliability analysis, are conventionally divided into two groups: level-3 methods which consider the whole geometry of the failure domain Df (defined by the values of the random variables for which the failure occurs) and level-2 methods which can deal only with idealized forms of the failure domain. Actually, the latters assume that Df is an hyperplane or hypersphere, so that the mathematical developments necessary to evaluate the probability of failure Pf became easy. In fact, these idealized forms are far from being the actual geometry of Df which is, generally, distorted with very irregular shape. So, the values of Pf deduced from the safety index, when level-2 methods are used, may be erroneous.

It is very difficult, and sometimes impossible, to perform exact integration of the probability density function of the random variables in the failure domain. Level-3 methods use then Monte Carlo simulations to evaluate the probability of failure. The required calculation time may be excessively important because Pf values are very small, in particular when dealing with ultimate limit state conditions. It appears then necessary to

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run other numerical procedures that are able to assess Pf values while requiring reduced computation time.

This paper describes a technique, the hypercone method, which combines the principles of level-2 and level-3 procedures. This hypercone method affords lower-bound and upper-bound values of Pf. It shows also the relative importance, in terms of contributions regarding the final value of Pf, of the various regions in the operating space.

## THEORETICAL ASPECTS OF THE HYPERCONE METHOD

Let $X=(X_1,..,X_i,....,X_n)$ be the random vector corresponding to the geometrical dimensions, the mechanical preperties of the constitutive materials and the applied loads. It must be transformed into a standardized gaussian vector $U=(U_1,U_2,...,U_n)$ where the random variables Ui are mutually independent.

If the basic random variables Xi are gaussian and independent, then Ui are deduced by the transformation:

$$U_i=(X_i-\mu_i)/\sigma_i \qquad \text{(1)}$$

where $\mu_i$ and $\sigma_i$ are, respectively, the mean and the standard deviation values of Xi.

If Xi are independent but not gaussian, they are transformed numerically through the relations:

$$\Phi(U_i)=F_i(X_i),\ i=1 \text{ to } n, \qquad \text{(2)}$$

where $\Phi(.)$ and $F_i(.)$ are, respectively, the cumulative probability functions of $U_i$ and $X_i$.

When the random variables Xi are statistically dependent, U vector is obtained by the means of Rosenblatt procedure, Melchers (1987), Leporati (1977)

For each given limit state, it is possible to associate a limit state function g(U) such as:

* $g(u) < 0$ if the limit state conditions are not reached
* $g(u) > 0$ if the limit state conditions are reached, then the failure occurs
* $g(u) = 0$ if the limit state conditions are strictly reached.

The operating random space can then be divided into several regions:

* Df = failure domain in which $g(u) < 0$
* Ds = safety domain in which $g(u) > 0$
* and the limit state surface where $g(u) = 0$.

As an illustration, let us consider a beam and study the ultimate limit state occuring by bending. Two random variables can be defined: the strength R and the action effect S, at the median cross section. The global random variable E=R-S can then describe the structure since:.

* $E < 0$ defines Df
* $E > 0$ defines Ds
* $E = 0$ defines the limit state surface.

In U-space, the hypercone method approaches the actual failure domain Df through sets of hyperconical sections, *see Fig 1*. The set containing Df affords an upper-bound value of Pf whereas the set contained in Df affords a lower-bound value for Pf, Mébarki (1990). The actual probability of failure is theoretically defined as:

$$Pf= \int \dots \int_{Df} f_U(u).du \quad \dots \dots \qquad (3)$$

where $f_U(u)$= the probability density function of U-vector and Df= failure domain in U-space

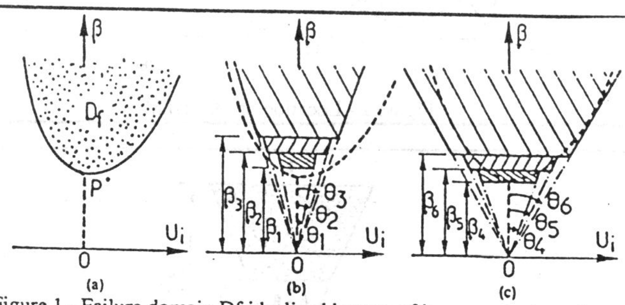



Figure 1 - Failure domain Df idealized by sets of hyperconical sections
(a)-Actual Df; (b)-Inscribed domain (c)-Circumscribed domain

**Main steps for application of the hypercone method,** Mébarki (1990):

1- Locate on the limit state surface, in U-space, the design point P* defined as the nearest point, on the limit state surface, to the origin O of the axes. The euclidian distance OP* is known as the safety index β, or Lind-Hasofer index. Its coordinates $u^*=(u_1^*,u_2^*,...,u_n^*)$ are solution of a minimization problem under equality constraints:

Minimize $f(u)$ where $f(u) = (u^T.u)^{1/2}$ ..................... (4)
under the equality constraint $g(u)=0$, $g(.)$ being the limit state function.

2- Once P* is located, consider along the straight line (OP*), succesive planes (Π*) that are perpendicular to (OP*). Locate then, in these planes, $M_n$ et $M_f$, belonging to the limit state surface and being, respectively, the nearest and the furthest points to (OP*). They will define the bases of the hyperconical sections having as respective half-apertures the angles ($OP^*,OM_n$) and ($OP^*,OM_f$) while they are, respectively, contained in Df or containing Df, *see Fig1 and 2*.

3- By summation of the integration values performed in the successive hyperconical sets, along (OP*), the lower-bound value $Pf_{min}$ and upper-bound value $Pf_{maj}$, are obtained by application of the basic formula corresponding to the integration inside an hyperconical section, *see Fig 3*:

$$Pf_c=V_c(\beta_1:\beta_2,\theta,n)=(2\pi)^{-1/2}\int_{\beta_1}^{\beta_2} \exp(-t^2/2).\chi^2_{n-1}(t^2.tg^2(\theta)).dt \quad \dots\dots\dots \quad (5)$$

where $\chi^2_{n-1}(.)$= Khi-square cumulative probability function with (n-1) degrees of freedom, θ=half-aperture of the hyperconical section, $\beta_1$ and $\beta_2$= distance of the bases, respectively lower and upper positions, defining the section.

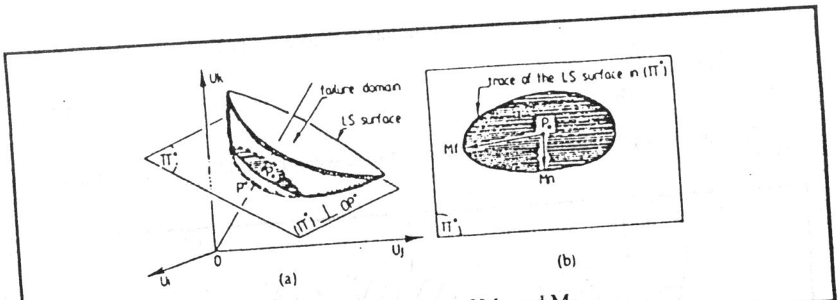



Figure 2 - Location of $M_n$ and $M_f$

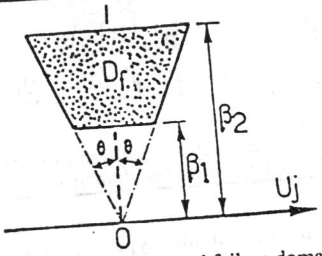



Figure 3- Hyperconical section as idealized failure domain (case *n*=2)

## APPLICATION OF THE HYPERCONE METHOD

### Definition of the RC structure

The structure under study, in this paper, is an RC beam under uniform vertical loads: a dead load G and a live load Q, Mébarki (1988a), *see Fig 4*. We assume that the geometrical dimensions, the mechanical strengths of the materials and the applied loads do not vary along the beam. The strength of the beam, against the ultimate limit state by bending, is then equal to the median cross-section strength.

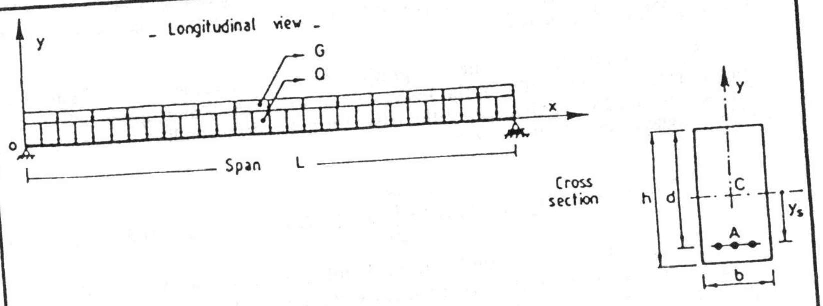



Figure 4- Definition of the structure

The reinforcement behaviour is described following BAEL rules (BAEL 1990):

$$\sigma = \min(|E.\varepsilon|, f_S) \quad \text{..............................} \quad (5),$$

where $\varepsilon$=strain, $\sigma$=stress, $E=2.1\times10^5$MPa=Young's modulus, fs=strength of the reinforcement.

The behaviour of the concrete under compression is defined following CEB-FIP recommendations (CEIB1979):

$$\sigma = (\mu.f_c)\ (k.\eta-\eta^2)/(1+(k-2).\eta) \quad \text{......................................} \quad (6),$$

$\eta = \varepsilon/\varepsilon'_o$, $k=Ec.\varepsilon'_o/(\mu.f_c)$, $\varepsilon'_o=\varepsilon_o(1+\Phi)$, $\Phi=(1.35\ G_k)/(1.35\ G_k + 1.5\ Q_k)$= loading coefficient

where $\mu$=0.85 (Rüsch's effect), $f_c$=strength under compression, $E_c$=Young's modulus, $\sigma$=stress corresponding to the strain $\varepsilon$, $\varepsilon_o=2\times10^{-3}$, $G_k$=2.5 kN/m$^2$ and $Q_k$=4.5 kN/m$^2$ are, respectively, the characteristical values of G and Q, BAEL (1990).

We selected seven basic random variables: b= width, h= depth, $y_s$= position of the reinforcement, $f_s$= strength of the reinforcement, $f_c$= strength of the concrete, loads G and Q. Their cumulative probability functions are described in Table 1, Mébarki (1988a). These random variables are assumed to be mutually independent.

**Table 1- Statistical description of the basic random variables**

| Parameter | Distribution | $X_k$ | Mean. $X_m$ | $C_V=\sigma/\mu$ C.O.V. | Truncatures $X_{min}$ | $X_{max}$ |
|---|---|---|---|---|---|---|
| b (cm) | Experimental | 25 | 25.1 | 0.02 | 23 | 29.5 |
| h (cm) | Experimental | 50 | 50.1 | 0.01 | 48 | 54.5 |
| $y_s$ (cm) | Experimental | 19 | 17.75 | 0.06 | 14.3 | 20.5 |
| $f_s$ (Mpa) | Gaussian | 400 | 470 | 0.11 | 341 | 645 |
| $f_c$ (Mpa) | Experimental | fc28=22 | 27.5 | 0.23 | 12 | 50 |
| G (kN/m$^2$) | Gaussian | 2.50 | Gm | 0.10 | 1.25 | 4.50 |
| Q (kN/m$^2$) | E1max | 4.50 | Qm | 0.35 | 0 | Free |

## Results

We have considered three possible values for the reinforcement ration: w=0.25 % and 0.5%=under-reinforced sections, w=1%=normally reinforced section and w=2%=over-reinforced section.

To locate the bases of the hyperconical sections, i.e. Mn and Mf, we have developped minimization techniques that do not need to any calculation of the limit state function derivatives, Mébarki (1990). Actually, these derivatives are not always available because the limit state functions are implicit.

The upper-bound and lower-bound values resulting from the hypercone method are given in Table 2.

**Table 2- Hypercone method results**

| w (%) | $Pf_{min}$ | $Pf_{maj}$ |
|---|---|---|
| 0.25 | 0.937E-4 | 0.176E-3 |
| 0.5 | 0.767E-4 | 0.142E-3 |
| 1 | 0.468E-4 | 0.113E-3 |
| 2 | 0.284E-4 | 0.278E-3 |

These collected results show that the hypercone method afford bound values of the probability of failure Pf that remain in reduced relative ratios (less than 10).

The relative ratio (Upper-bound value/Lower-boun value) is higher for over-reinforced sections in which the failure is mainly governed by the behaviour of the concrete. For this kind of structure, it is obvious that the failure is sensitive to the concrete quality (mean and standard deviation values) since the reinforcement remains in the elastic part of its behaviour.

To visualize the limit state surface, we have neglected the random nature of the width b, the depth h and the position ys of the reinforcement. They are assumed to take deterministic values (equal to their characteristical values afforded in Table 1) while the strengths fc, fs and the loads G and Q remain random variables.

The loads G and Q are replaced by their global effect S=G+Q which distribution is calculated numerically from those of G and Q, Mébarki (1990). The shapes of the limit state surfaces are shown in *Fig 5*.

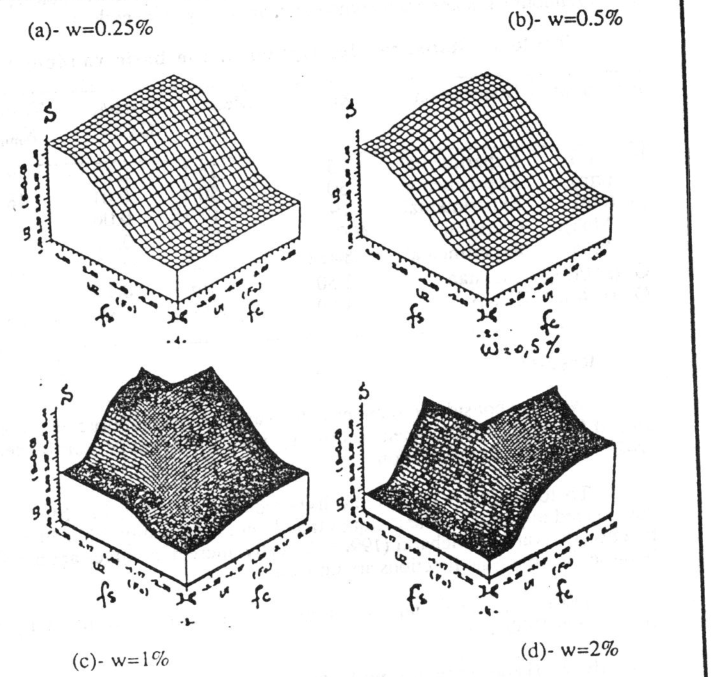



Figure 5- Shapes of the limit state surfaces

These figures show the sensitivity of the limit state surface to:

* the reinforcement quality for under-reinforced sections because the steel reaches the plastic part of its behaviour while the strain remain small in the concrete

*the concrete quality for over-reinforced sections because the steel remain in the elastic part ot its behaviour while the concrete is under excessive compression. The probability of failure takes large values for this kind of structures. As the risk of failure is important, a great attention must then be devoted to the quality of the concrete.

This sensitivity remains equivalent regarding the reinforcement or concrete qualities for normally-reinforced structures.

## LEVEL-2 METHODS and SAFETY INDEX β

### Theoretical principles

To the safety index β, may be associated an operational value, $Pf_L$, of the probability of failure:

$$Pf_L=\Phi(-\beta) \quad \text{.............................................} \quad (7)$$

where Φ(.)= cumulative distribution function of the standardized gaussian distribution. This association assumed that the limit state surface is an hyperplane perpendicular to OP*. It is obvious that this is an idealized case that, in practice, may differ greatly from the real shape of the actual limit state surface.

### Results

The collected results are given in Table 3.

**Table 3- Values of the safety index β .**

| w (%) | Index β | $Pf_L=\Phi(-\beta)$ |
|---|---|---|
| 0.5 | 3.55 | 0.118E-3 |
| 1 | 3.64 | 0.754E-4 |
| 2 | 3.49 | 0.166E-3 |

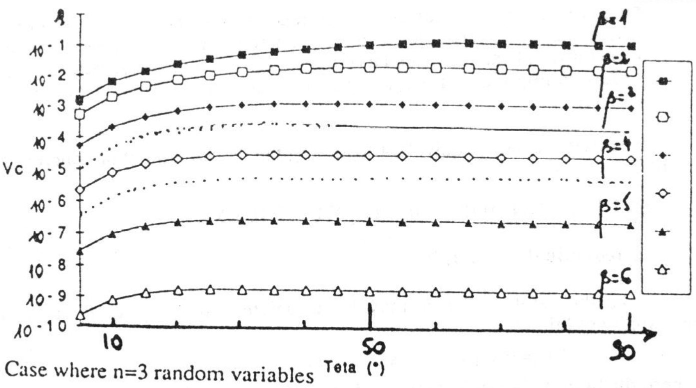
These operational values of Pf are in accordance with the hypercone method since they fall within the interval $[Pf_{min} \div Pf_{maj}]$. The hypothesis stating that the limit state surface is planar seems to be acceptable. This means that there is a mutual equilibrium between the convex and concave parts of the failure domain. In fact, Fig 5 shows that this surface is not planar. The results given in Fig 6 mean that a restricted region of the operating space has the major contribution in the final value of Pf. Actually, for n=3 random variables and safety index ranging from 3 to 4, the "probabilistic weight" is concentrated around the design point P* in an hypercone section having an aperture of about (2x15°). The other regions, that are far from P*, have very small influence on Pf. This is why the linear form of the failure domain is an acceptable hypothesis, for the structures considered herein. The region of importance may differ for other values of the number of random variables involved, for instance n=10.

## LEVEL-3 METHOD: Monte Carlo simulations

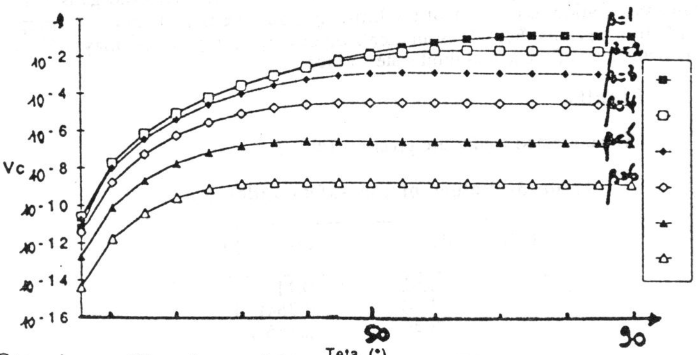
### Theoretical principles

When the Monte Carlo simulations are used, the probability of failure can be defined as:

$$Pf=Prob(g(x)<0)=n_1/N_{sim} \quad \text{.................................} \quad (8)$$




(a)- Case where n=3 random variables




(b)- Case where n=10 random variables

Figure 6- Integrals calculated for hyperconical sections

where $N_{sim}$=number of simulations performed, $n_1$=number of times where the failure occurs. Doing so, a large number of simulations is required, greater than $n_1/Pf$, leading to excessive computation time. To reduce this number, a technique which separates the random variables into two groups may be ran, Mébarki (1990). Actually, we considered a first group constituted by b, h, $y_s$, $f_c$ and $f_S$, and a second group involving G and Q. The probability of failure is then:

$$Pf_{mc}=(1/N_{sim}).\ \sum_{k=1,N_{sim}} (1-F_s(r^{(k)})) \qquad \text{.............................} \qquad (9)$$

where $F_s(s)$=Prob(G+Q <s)=cumulative distribution function of the loads effect S (S=G+Q) which values are calculated numerically, $r^{(k)}$=strength of the structure obtained at the $k^{th}$ simulation.

## Results

The results obtained show that $N_{sim}$=200 simulations is a sufficient number to obtain a correct value of Pf. The values are given in Table 4

Table 4- Pf values obtained by simulations

| w (%) | $Pf_{mc}$ |
|---|---|
| 0.5 | 0.105E-3 |
| 1 | 0.725E-4 |
| 2 | 0.143E-3 |

This technique separating the random variables affords precise values for Pf, in comparison to those obtained with the hypercone method, without running a large number of simulations: $N_{sim}$=200 is sufficient whereas the required number, in accordance with eq.(8), would have been $n_1$>10000.

## CONCLUSIONS

The results obtained in this paper show that:

- Monte Carlo simulations may require a reduced number of simulations ($N_{sim}$=200) if the basic random variables are adequately separated,

- The hypothesis stating a linear form of the limit state surface, in order to deduce operational values of Pf from the safety index, affords results that are in accordance with more sophisticated methods (Monte carlo simulations and the hypercone method). This is explained by the fact that the probability of failure has a great sensitivity to the region located near the design point P*: an hypercone section having an aperture almost equal to (2x15°),

- The hypercone method gives lower and upper bounds values of Pf that are in small relative ratios: less that 10. This technique constitutes a helpful tool for the accuracy analysis of the usual methods used in reliability analysis.

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