NUMERICAL CALCULATION OF FRACTURE MECHANICAL WEIGHT FUNCTIONS FOR CRACKS IN FINITE BODIES

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Weight functions for two- and three-dimensional solids with cracks are calculated by boundary element methods (BEM). In the two-dimensional case most general elastic anisotropy (triclinic) is considered. Two novel approaches are introduced to derive weight functions for cracks in finite bodies. The first method provides the weight function by superposition of the Bueckner fundamental field for the crack in the infinite region and a proper finite body correction. The second method is based on a regular numerical approximation of the Bueckner singularity. The advantages of the BEM are used consistently within these approaches.

INTRODUCTION

In the framework of linear elastic fracture mechanics the K-factor concept is used for the assessment of cracks in complex structures. Consider an arbitrarily shaped crack in a finite or infinite region Ω. A local coordinate system (x_1, x_2, x_3) at the crack front is used with x_2 normal to the crack faces and x_1 normal to x_2 and the crack front. Arbitrary loadings induce a singularity at the crack tip such that the stress components ahead of the tip on $x_2=0$ vary like

$$\sigma_{n2} \sim K_n / (2 \cdot \pi \cdot x_1) \qquad (1)$$

where K_1, K_2 and K_3 are the mode II, mode I and mode III stress intensity factors, respectively.

The field equations for a linear elastic solid are:

$$\sigma_{ij,j} = -f_i \qquad (2)$$

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \qquad (3)$$

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$$\varepsilon^G_{ij} = \frac{1}{2}\cdot(u_{i,j} + u_{j,i}) \tag{4}$$

$$\varepsilon^G_{ij} = \varepsilon_{ij} + \varepsilon^a_{ij} \tag{5}$$

The total strain ε^G is the sum of the elastic and the anelastic strain. In the whole paper the summation convention is used.

The fracture mechanical weight function $h_{ni}(R,Q)$ is defined by the equation

$$K_n(Q) = \int_\Omega h_{ni}(R,Q)\ f_i(R)\ dV \tag{6}$$

Here ε^a is assumed to vanish identically in the whole region Ω.

When the forces only act as boundary tractions $t_i(R) = \sigma_{ij}n_j$, n being the outward normal vector in R, the volume integral in equation (6) reduces to an integral over the boundary Γ of the volume:

$$K_n(Q) = \int_\Gamma h_{ni}(R,Q)\ t_i(R)\ dA \tag{7}$$

If the stress field is induced only by anelastic strains (no boundary tractions and no volume forces act) application of Betti's theorem yields the following equation for the stress intensity factors (Rice (1)):

$$K_n(Q) = \int_\Omega L_{nij}(R,Q)\ \varepsilon^G_{ij}(R)\ dV \tag{8}$$

$$L_{nij} = \frac{1}{2}\, C_{ijkl}(h_{nk,l} + h_{nl,k}) \tag{9}$$

Equation (8) can be transformed into

$$K_n(Q) = \int_\Gamma h_{ni}(R,Q)\ t^a_i(R)\ dA - \int_\Omega h_{ni}(R,Q)\ f^a_i(R)\ dV \tag{10}$$

$$t^a_i = C_{ijkl}\varepsilon^a_{kl}n_j \qquad f^a_i = (C_{ijkl}\varepsilon^a_{kl})_{,j} \tag{11}$$

Equations (6-10) mean that once the weight function for a given geometry is known the K-factors can be calculated by a simple quadrature. In the case of thermal loading equation (10) allows to calculate the stress intensity factors only from the given temperature distribution and their derivatives.

An elegant approach to weight functions is based on Bueckner's (2) fundamental field (BFF). The BFF of mode n is the singular displacement field u_n induced by a pair of point forces P_n acting in opposite directions each at one crack face, a small distance c from the crack tip when shrinking c to zero under the condition $P_n \cdot c^{1/2}$= constant = B_n. If B_n and the direction of the forces are properly chosen, u_n turns out to be the weight function h_n. BFFs for some types of cracks in the infinite region can be given analytically.

TWO-DIMENSIONAL FUNDAMENTAL FIELDS IN TRICLINIC SOLIDS

The displacement field has to satisfy equation (2) which can be recast in the form

$$D_{ij}u_j = 0 \quad \text{with } D_{ij} = D_{ij}(\partial_1,\partial_2) = C_{ijkl}\partial_k\partial_l \;;\; \partial_i = \frac{d}{dx_i} \tag{12}$$

In the most general case of elastic anisotropy, 21 components of C are independent of each other. When plane deformation is assumed (all field quantities only depend on x_1 and x_2) the number of independend components of C reduces to 15. Stroh (3) derived a general solution of (12) in the form

$$u_k = \sum_{l=1}^{3} \mathrm{Re}\{ A_{kl} g_l(z_l)\} \;;\; z_l = x_1 + p_l x_2 \tag{13}$$

where g_l is an arbitrary function and A_{kl} and p_l are complex. Equation (12) is satisfied if the condition

$$D_{jk}(1,p_l)A_{kl} = 0 \tag{14}$$

is met. For nontrivial A_{kl} the determinant det(D) of the matrix $D_{jk}(1,p_l)$ has to vanish. This leads to a sextic equation for the roots p which occur in complex conjugate pairs. The corresponding eigenvectors are (A_{1l},A_{2l},A_{3l}). In general, the crack extension force G can be written as

$$G = K_i L_{ij} K_j \tag{15}$$

The BFF for a semi-infinite crack can be represented in the following form (Sham and Zhou (4))

$$h_{ni} = -\frac{L_{ij}}{2(2\pi)^{1/2}} \sum_{l=1}^{3} \mathrm{Re}\{ T^{-1}_{jl} A_{nl} z_l^{-1/2}\} \tag{16}$$

with T^{-1} and L^{-1} being the matrices invers to T and L and

$$L_{ij} = -\frac{1}{2} \sum_{l=1}^{3} \mathrm{Im}\{A_{il} T^{-1}_{jl}\} \tag{17}$$

$$T_{kl} = B_{k2l} = B_{2kl} \;;\; B_{knl} = (C_{knj1} + p_l C_{knj2})A_{jl} \tag{18}$$

Sham and Zhou gave an analytical expression of the eigenvectors for monoclinic materials, where the (x_1,x_2)-plane is a plane of symmetry. Maschke (Busch et.al.(5)) derived an explicit expression for the eigenvectors which is valid even in the most general triclinic case. The condition det(D) = 0 can be written in the form

$$\delta_{ik} \det(D) = D_{ij}(1,p_l) \cdot D^*_{jk}(1,p_l) = 0 \tag{19}$$

where D^* is the matrix adjoint to D. Therefore the eigenvectors can be expressed as

$$A_{jl} = D^*_{jk}(1,p_l) \tag{20}$$

with arbitrary but fixed k = 1,2 or 3. Maschke (6) also derived closed form solutions for the weight functions for a finite and two semi-infinite collinear cracks for triclinic materials.

WEIGHT FUNCTIONS FOR CRACKS IN FINITE BODIES

Since analytical weight function solutions for cracks in finite bodies are not available, numerical methods must be applied for their calculation. While at first FEM was used, later on BEM became more and more attractive. The Bueckner singularity was incorporated into the numerical model by cutting out a small region around the crack tip, or by use of a power series expansion to approximate the BFF for the crack (see e.g. Rooke et.al. (7), Aliabadi et.al. (8)). While the first leads to a high effort in discretization, the latter is an approximation valid exclusively if there is only one crack tip to be considered. In order to determine weight functions for arbitrarily shaped finite bodies two novel numerical methods were developed.

Weight functions for special cracks by superposition

Although the strain energy connected with the BFF is unbounded the superposition technique can be applied to compute weight functions for cracks in finite bodies. The weight function for the crack in the finite body is obtained by superposition of the fundamental field of the considered crack geometry in the infinite region, and a regular solution of a crack problem which guarantees zero boundary tractions (see (5)). As an example the weight functions for a circle with one edge crack, two edge cracks and a centre crack (Figure 1) were calculated. The K-factors are evaluated from equation (7) for boundary tractions producing a single fracture mode K_j only (for details see (6)). The evaluation of equation (7) with the weight function h_n then must yield $K_n = \delta_{jn}$.

case	a)			b)			c)		
j	K_1	K_2	K_3	K_1	K_2	K_3	K_1	K_2	K_3
1	1.0002	0.0003	-.0002	0.9942	-.0034	-.0007	1.0072	0.0039	-.0010
2	-.0009	0.9996	0.0001	0.0019	0.9989	0.0010	0.0003	0.9922	-.0005
3	-.0001	0.0004	0.9999	-.0033	0.0011	1.0007	0.0008	-.0010	0.9994

As a three-dimensional example the weight function for the tensile specimen illustrated in Figure 3 was calculated. The normalized K-factor distribution for constant shear loading along the crack front obtained by weight function evaluation are compared with direct BEM-results (Figure 4).

Weight functions for arbitrary cracks by regular approximation

The weight function is generated by two point forces acting at the crack tip. An approximation of the weight function is obtained if the point forces are replaced by regular boundary tractions acting in a small region on the crack faces closely to the crack tip. In our BEM-codes special crack tip elements are implemented. In the crack tip elements on the crack faces tractions are chosen in such a manner that the displacements caused by the tractions are a good approximation of the weight function in that sense that equations (6-10) are fullfilled for a great variety of loads. This condition yields to 3 or 8 equations for the determination of the tractions in the two- or three-dimensional case, respectively. As an example the weight function h_2 for a tensile specimen of isotropic material was calculated for the four different crack configurations illustrated in Figure 2. The K-factor K_2 was evaluated for constant tension on the upper and lower side. Additional to b) the inverse case of a short crack was considered (point b2)).

case	a)	b)	b2)	c)	d)
K_{dir}	11.45	40.71	4.39	10.90	11.44
K_{WFM}	11.40	39.81	4.41	10.81	11.45

The method was also applied to the 3D tension specimen cited below. The results are plotted in figure 4 as well.

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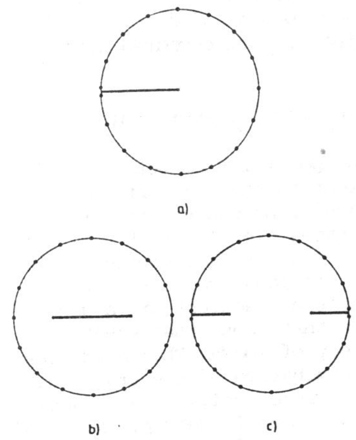

Figure 1 2D-example for superposition method

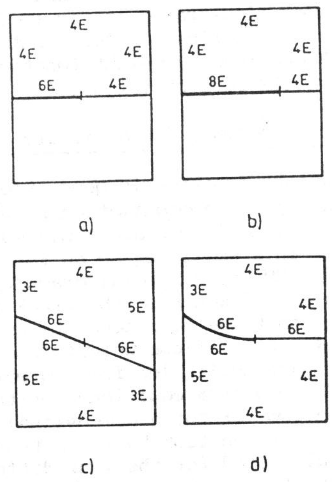

Figure 2 2D-example for approximation method

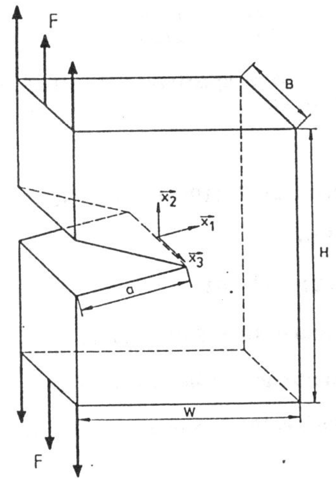

Figure 3 Tensile specimen as 3D-example

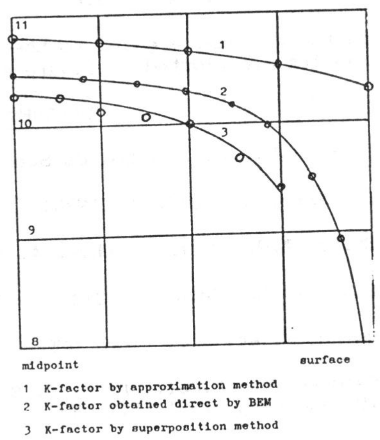

Figure 4 K-factor distribution along crack front