LOCALLY AND FULLY COUPLED APPROACHES TO THE RUPTURE OF BRITTLE MATERIALS

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Two different methods are analyzed to study the failure of structures made of brittle materials. Generally, the cause of rupture is due to initial flaws randomly distributed within the structure. The two following methods consider the statistical aspect of the flaw distribution.

The fully coupled method consists in taking account of the flaw distribution when the stress field is computed. This method is entirely numerical. The locally coupled method neglects the interaction between defects, and therefore can be used in a post-processor strategy. This method is either numerical or analytical, and can be extended to cyclic cases.

INTRODUCTION

Initial flaws are the main cause of the failure by fracture of structures made of brittle materials. In most of these materials, the flaws are randomly distributed, and therefore one needs a statistical approach to assess the failure probability. A weakest link assumption (see Weibull (1), Freudenthal (2)) is used to determine the global failure condition.

Two approaches can be utilized to compute the failure probability of a structure. The first one, referred to as *fully coupled*, models the initial flaws when the stress field is computed, and takes account in one way of their interactions. The second one, referred to as *locally coupled*, neglects the interaction between flaws, and thus can be implemented as a post-processor to a finite element code. This latter approach enables us to derive a closed-form relationship between the flaw distribution and the failure probability in the case of monotonic loadings. By studying the flaw size evolution and the corresponding flaw distribution, one can generalize the previous results to cyclic loadings.

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MECHANICAL MODELING OF A LINK

A brittle material can be regarded as a series of links. The failure of one of them leads to the complete failure of the structure. The initial flaws, cause of the rupture, can be modelled by an initial value of a scalar damage variable, D_0, assumed to be constant over one link. The value of D_0 is dependent upon the ratio a/L, where a denotes the flaw size, and L the length of a cubic link of volume V_0. It is worth noting that the size of the link, L, is at least equal to the maximum flaw size, a_M, and at most equal to the average distance between two adjacent flaws. When D_0 is equal to zero, the link contains no flaw. Conversely, when D_0 is different from zero, the link contains a flaw, and the mechanical properties are degraded.

Because of the presence of a flaw, the resisting area is reduced and the effective stress tensor $\tilde{\sigma}$ is utilized instead of the Cauchy stress tensor: $\tilde{\sigma} = \sigma / (1-D_0)$. Since the failure is mainly driven by tensile stresses, the damage equivalent stress (see Lemaitre and Chaboche (3), Hild and Marquis (4)) is referring to the positive stress tensor σ^+

$$\sigma_r^+ = \sqrt{\frac{2}{3}(1+\nu)\left(\sigma_H^+\right)^2 + 3(1-2\nu)\left(\sigma_{eq}^+\right)^2} \qquad (1)$$

where ν is the Poisson's ratio, σ_H^+ and σ_{eq}^+ are the two first invariants of the positive stress defined as

$$\sigma^+ = \sum_{I=1}^{3} <\sigma_I> V_I \otimes V_I \qquad (2)$$

where $\otimes$ is the tensorial product, σ_I are the principal stresses, V_I are the corresponding normalized eigen vectors, and $<.>$ denotes the McCauley brackets. The failure criterion is referring to the effective damage equivalent stress

$$\sigma_r^+ / (1-D_0) = \sigma_M \qquad (3)$$

where σ_M is a material parameter. Equ. (3) can be rewritten as

$$Y^+ = Y_c \qquad (4)$$

where Y^+ is the damage energy density release rate (see Ref. (3)) referring to the positive stress tensor, and Y_c is a material parameter.

When the failure criterion is satisfied, the damage variable evolves and leads to the failure of the link, and to the failure of the structure as well, since a weakest link assumption is used.

The cumulative failure probability, P_{F0}, of a link of volume V_0 is the probability of finding a flaw greater than or equal to a critical flaw size, $a_c(\sigma_r)$. The critical flaw size, $a_c(\sigma_r)$, depends directly upon the load level characterized by the equivalent stress σ_r, upon the rupture criterion, and upon the relationship between D_0 and a/L. The cumulative failure probability, P_{F0}, can be related to the initial flaw distribution, f_0, by

$$P_{F0} = \int_{a_c(\sigma_r)}^{a_M} f_0(a)\, da \qquad (5)$$

Expression (5) can be extended to cyclic loadings. In this case, the flaw distribution, f, evolves with the number of cycles, N, since the flaw size evolves with the number of cycles, and is denoted by f_N. The evolution can for instance be given by a Paris law, and therefore the function f_N depends upon the maximum equivalent stress over a cycle σ: $f_N(a,\sigma)$. Let us introduce a function ψ such that $a_0 = \psi(a_N,\sigma,N)$, which depends upon the evolution law of the flaw size (Hild and Roux (6)), where a_0 is the initial flaw size, and a_N is the flaw size after N cycles. The evolution is supposed to be deterministic, thus the probability of finding a flaw of size a after N cycles is equal to the probability of finding an initial flaw of size $\psi(a_N,\sigma,N)$. Since it is assumed that no new crack initiates, the function f_N can be related to the initial flaw distribution f_0 by

$$f_N(a,N) = f_0[\psi(a_N,\sigma,N)]\, \frac{\partial\psi}{\partial a} \qquad (6)$$

where the coefficient $\partial\psi/\partial a$ comes from the change of measure (from da to $d\psi$). By means of Equ. (5), and Equ. (6) a relationship between the initial flaw distribution and the cumulative failure probability of a link can be derived in the case of cyclic loadings

$$P_{F0} = \int_{\psi(a_c(\sigma),\sigma,N)}^{a_M} f_0(a)\, da \qquad (7)$$

For monotonic loadings, the same expression holds since $a_c(\sigma) = \psi(a_c(\sigma),\sigma,N=0)$. Equ. (7) gives a unified expression of the cumulative failure probability of a link in the case of brittle failure under monotonic and cyclic loadings.

FULLY COUPLED APPROACH

This approach takes account of the the influence of the flaws upon the stress field, and necessitates as many calculations as flaw configurations (see Fig. 1). Indeed the flaw distribution (flaw size distribution, and distance between two adjacent flaws) is realized on average by several random selections using a Monte-Carlo method. Therefore the interactions between flaws can be very different from one realization to another and lead to different load levels at failure. This method is entirely numerical and is carried out by using a standard finite element code ABAQUS (5). The advantage of this method is to take account of the flaw interaction. The drawback of this method is that many computations are needed to be able to asses the failure probability of structures: in practice, at least 50 computations are done per problem.

LOCALLY COUPLED APPROACH

An alternative method to the previous one is to assume that the flaw interactions are small. In this case, we neglect the interactions and only one computation is needed in order to get the stress field in the structure free of any flaw (see Fig. 2). The failure analysis is performed by using a post-processor in which different defect configurations are analyzed. This can be done either by using again a Monte-Carlo method (i.e. a numerical simulation), or by using a closed form expression.

When the interactions are neglected, an independence of events hypothesis applies. The cumulative failure probability, P_F, of a structure Ω of volume V can be related to the initial flaw distribution, f_0, by

$$P_F = 1 - \exp\left[-\frac{1}{V_0} \int_{\Omega} \ln\left\{ 1 - \int_{\psi(a_c(\sigma),\sigma,N)}^{a_M} f_0(a)\, da \right\} \right] \qquad (8)$$

This expression is valid in the case of monotonic and cyclic loadings.

GENERAL PROPERTIES

The fully coupled and the locally coupled approaches are able to model the size effect observed in brittle materials. This is directly due to the flaw distribution. Indeed, as the volume increases, the probability of finding a larger flaw increases, hence the failure probability increases.

The two approaches are also able to model the stress heterogeneity effect. The more heterogeneous the stress field, the lower the probability of finding a critical flaw in the most loaded area, therefore the lower the failure probability (see Ref. (4)).

It is worth noting that these two effects exist under monotonic *and* cyclic conditions. These two properties are directly related to the statistical nature of the flaw distribution.

Using the locally coupled approach, we can get a cumulative failure probability, P_{F0}, taking the form of a three parameter Weibull law (Ref. (1)). We assume that the damage variable, D_0, is related to the ratio a/L by a positive strictly increasing C^1 function h: $D_0 = h(a/L)$. The critical flaw size, $a_c(\sigma_r)$, is therefore related to the equivalent stress, σ_r, by: $\sigma_r = \sigma_M\,[1 - h(a_c)/L)]$. The threshold stress,

σ_u, defined as the stress under which the failure probabilty has a zero value, is given by

$$\sigma_u = \sigma_M \, [1-h(a_M/L)] \quad (9)$$

If we assume that for a close to a_M the function f_0 is equivalent to $k(a_M - a)^\beta$, with $k>0$, and $\beta>0$, one can get a relationship between the shape parameter, m, the exponent β given by

$$m = \beta + 1 \quad (10)$$

and an expression of the scale parameter, σ_0,

$$\sigma_0 = \sigma_M \, h'(a_M/L) \left[\frac{\beta+1}{k}\right]^{1/(\beta+1)} \quad (11)$$

These three last equations show that a Weibull law can be related to a initial flaw distribution (see Ref. (4)), and show that parameters related to the flaw distribution can be linked with parameters related to macroscopic quantities. The same kind of results can be found by using Linear Elastic Fracture Mechanics (see Ref. (4)).

CONCLUSION

The two proposed approaches take explicitly account of the flaw distribution in the computation of the failure probability. For the locally coupled approach, Equ. (5), valid in the case of monotonic loadings, can be generalized to Equ. (7) by writing the evolution of the flaw distribution with the number of applied cycles. This allows us to present a unified framework for brittle failure under monotonic and cyclic loadings, taking account of the initial flaw distribution within the material. For the fully coupled approach, the extension to cyclic loadings is not considered, since it will involve to much computation time.

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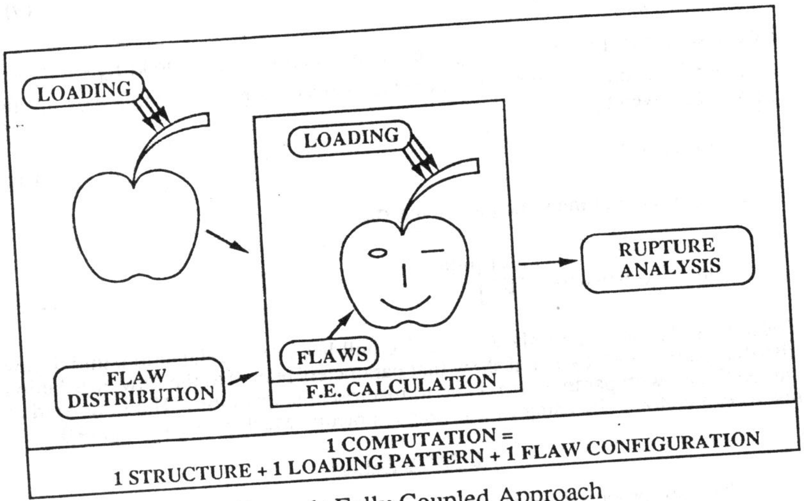

Figure 1: Fully Coupled Approach

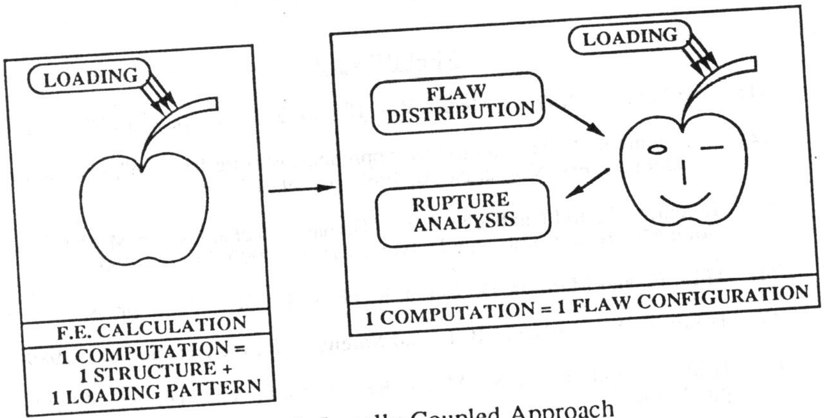

Figure 2: Locally Coupled Approach