FRACTURE MECHANICAL MARKOV CHAIN FATIGUE MODEL

Lise Gansted *, Rune Brincker * and Lars Pilegaard Hansen *

On the basis of the B-model developed in Bogdanoff and Kozin (1) a new numerical model incorporating the physical knowledge of fatigue crack propagation is developed. The model is based on the assumption that the crack propagation process can be described by a discrete space Markov theory. The model is applicable to deterministic as well as to random loading. Once the model parameters for a given material have been determined, the results can be used for any structure as soon as the geometrical function is known.

INTRODUCTION

The Fracture Mechanical Markov Chain Fatigue Model (FMF-model) is based on the B-model, see Bogdanoff and Kozin (1), i.e. on the assumption that the crack propagation process can be described by a discrete space Markov theory. The discrete time is measured in number of so-called duty cycles ($x = 1, 2, \ldots,$ number of DCs) each consisting of a number of load cycles (λ), and the crack progress is described by a series of discrete damage states ($d = 0, 1, 2, \ldots, b$), where b corresponds to failure.

The damage accumulation is considered as a stochastic process in which the possibility of damage accumulation is present each time the structure has experienced a duty cycle. It is assumed that the increment of damage at the end of the DC only depends on the DC itself and the state of damage present at the start of the DC, i.e. the history has no influence. The damage only increases by one unit at a time.

The damage accumulation process is completely described by its transition matrix (one for each duty cycle) and by the initial conditions.

* Department of Building Technology and Structural Engineering, University of Aalborg

Once the model parameters are determined, the state of damage in the given structure is available at any time This means that all statistical information about the damage process can be represented by the model.

The model parameters are determined using a fracture mechanical point of view. This is the main point where the FMF-model differ from the B-model in which the parameters are determined solely from empirical data.

The crack length, a, is used as a damage measure which is advantageous since a is a quantity that easily can be observed. The damage is assumed to progress in steps of the length δa. Thus, the j th state of damage can be defined as

$$a_j = a_0 + j\ \delta a \qquad j = 0, 1, 2, \ldots, b \tag{1}$$

In agreement with the Markov assumption damage is only accumulated when the crack propagates. As long as the crack remains in a given state, the same trial is repeated each time a DC is applied. This means, that in a given crack state, $a = a_j$, the propagation of the crack can be modelled by a Bernoulli random variable

At each crack position, given by (1), the random variable δN_j, which is the number of duty cycles applied to propagate the crack one step from a_j to $a_j + \delta a$, is considered. It is assumed that a_0 is constant so that the δN_j-values express the properties of the material. δN_j is modelled as a geometric distribution with the expected value and the variance given as the first and second moment, respectively. Cf. Benjamin and Cornell (2)

$$\mathrm{E}[\delta N_j] = \sum_{\delta n_j = 0}^{\infty} \delta n_j \,(1 - q_j)^{\delta n_j - 1} = \frac{1}{q_j} \tag{2}$$

$$\mathrm{Var}[\delta N_j] = \mathrm{E}[\delta N_j^2] - (\mathrm{E}[\delta N_j])^2 = \frac{1 - q_j}{q_j^2} \tag{3}$$

The quantity q_j is known as the transition probability. The crack growth problem is then reduced to the determination of the transition probability which is a function of the stress intensity factor range which is a function of the crack state, i.e. $q_j = q(\Delta K_j) = q(\Delta K(a_j))$.

The damage process is described by Paris law

$$\frac{da}{dN} = C(\Delta K)^m \tag{4}$$

The crack growth rate can be estimated in different ways depending on the definition of the slope of the sample curves. It is assumed that the number of duty cycles are observed for fixed values of the crack length and that the duty cycles are equal. In the FMF-model, the crack growth rate is defined as the step length divided with the mean value of number of duty cycles, $\mathrm{E}[\delta N_j]$, spent in a crack state, i.e,

$$\frac{\delta a}{\mathrm{E}[\delta N_j]} = \lambda\, C\, (\Delta K_j)^m \tag{5}$$

Notice that the Paris law has not become stochastic, all quantities in (5) are deterministic. (δN_j is a stochastic variable, but it is $\mathrm{E}[\delta N_j]$ which is used in (5)).

Equation (2) and (5) gives

$$q_j = \frac{\lambda\, C}{\delta a}(\Delta K(a_0 + j\delta a))^m = \frac{\lambda\, C}{\delta a}(\Delta K(a_j))^m \tag{6}$$

The total number of duty cycles applied to the structure to propagate the crack to the crack length a_j is

$$N_j = \sum_{k=0}^{j-1} \delta N_k \tag{7}$$

Because the random variable is a sum of independent random variables, the expected value of number of duty cycles is

$$\mathrm{E}[N_j] = \mathrm{E}\left[\sum_{k=0}^{j-1} \delta N_k\right] = \sum_{k=0}^{j-1} \frac{1}{q_k} \simeq \frac{1}{\lambda\, C} \int_{a_0}^{a_j} (\Delta K(a))^{-m}\, da \tag{8}$$

where the sum is approximated with an integral.

The variance of a sum of independent variables is given as, see ref. (2)

$$\mathrm{Var}[N_j] = \sum_{k=0}^{j-1} \mathrm{Var}[\delta N_k] = \sum_{k=0}^{j-1} \frac{1-q_k}{q_k^2} = \frac{\delta a}{\lambda^2\, C^2}\, [f(a) - g(a)] \tag{9}$$

where the sum is approximated with an integral and where

$$f(a) = \int_{a_0}^{a_j} (\Delta K(a))^{-2m}\, da \tag{10}$$

$$g(a) = \frac{\lambda\, C}{\delta a} \int_{a_0}^{a_j} (\Delta K(a))^{-m}\, da \tag{11}$$

The only unknown quantity left is the step length δa, which can be estimated if the variance of N_j is known from experiments.

In the case of random load, the stress intensity factor will also vary randomly and the well-known effects of acceleration and retardation might occur. The FMF-model itself does not take into account these interaction effects. This can be done using a crack closure model when ΔK is calculated, e.g. by introducing the effective stress intensity factor range,

$$\Delta K_{eff} = \Delta\sigma_{eff} F \sqrt{\pi a} = (\sigma_{max} - \sigma_{cl}) F \sqrt{\pi a} \tag{12}$$

Thus, the FMF-model is also available if the load is random. The only changes to be done are replacing ΔK with ΔK_{eff} in (4), (5), (6), (8), (10) and in (11).

SYMBOLS USED

a_0	=	initial crack length
a_b	=	failure crack length
a_c	=	critical crack length
δa	=	crack length increment
b	=	failure state
d	=	damage state
δN	=	increase in numbers of duty cycles
q	=	transition probability
x	=	discrete time
λ	=	numbers of load cycles in one duty cycle
σ_{cl}	=	crack closure stress

REFERENCES

Bogdanoff, J.L. and Kozin, F. "Probabilistic Models of Cumulative Damage" John Wiley & Sons, Inc. 1985

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