

FILTER TECHNIQUE FOR STOCHASTIC CRACK GROWTH

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Based on the concepts of fracture mechanics various probabilistic models have been developed to predict crack growth. Each of them, however, present significant inconveniences.

A model, for fatigue crack propagation under stochastic loads, based on a continuous Markovian scheme, is achieved by an adequate filter technique. It was recently proposed to solve most of the inconveniences of the previous approaches and is discussed in the paper. A numerical example illustrates the accuracy of the approach.

INTRODUCTION

The variability of the fatigue strength of material and the uncertain nature of service load suggest the adoption of a probabilistic approach to the analysis of fatigue problems. Several probabilistic models have been developed to predict crack growth in the framework of fracture mechanics. The following three main categories can be identified (1):

a) approaches which make use of an equivalent or characteristic loading for studying the crack propagation under stochastic amplitudes;

b) approaches which make use of continuous or discrete Markov models. Here one studies the crack propagation under stochastic loading and/or in the presence of randomness of the material parameters;

c) approaches which make use of the central limit theorem of probability. The crack length after  $n$  load cycles is regarded as the sum of random events.

These models, unfortunately, present several inconveniences. Characteristic loading models cannot yield sufficient information about a stochastic process

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since they adopt a single parameter to describe the whole influence of the load randomness on the crack growth. Continuous Markov idealizations give rise to non zero probability of negative crack increments when the solution is pursued by the Stratonovich average method (3). Discrete Markov models require a large data base for crack size versus cycles in order to estimate large transition matrix. Moreover, they cannot be used when one is interested in the probability of failure after a very large number of load cycles. In order to avoid the inconveniences listed above, a new model for fatigue crack propagation under stochastic loads is presented. It is based on a continuous Markov scheme. An adequate filter technique (2) is introduced to build this new model.

GOVERNING RELATIONS AND PROPOSED MODEL

The mathematical equation for the crack propagation rate under cyclic loads is:

$$da(t)/dt = Q(K, \Delta K, s, R, a(t)) \dots \dots \dots (1)$$

In Eq.(1)  $a(t)$  is the crack size at time  $t$ ,  $Q$  is a non negative function of the crack size, the stress intensity factor  $K$ , the stress intensity range  $\Delta K$ , the stress amplitude  $s$  and the stress ratio  $R = s_{min}/s_{max}$ . Eq.(1) can be "randomized" as follows (5):

$$d\underline{a}(t)/dt = Q(K, \Delta K, s, R, \underline{a}(t)) \underline{X}(t) \dots \dots \dots (2)$$

where the factor  $\underline{X}(t)$  is a non negative stationary random process (underlined symbols denotes random variables).

In order to take into account the randomness of the load process one can use a "characteristic load amplitude" to replace the stress amplitude  $s$  in Eq.(2). In particular  $s_{rms}$  is defined as the root mean square of the stress amplitude. Eq.(2) can be rewritten as:

$$d\underline{a}(t)/dt = Q(K_{rms}, s_{rms}, R, \underline{a}(t)) \underline{X}(t) \dots \dots \dots (3)$$

where  $K_{rms}$  is the root mean square of the stress intensity factor. Here the random factor  $\underline{X}(t)$  represents the combined effect of further contributions which result in effect is changing the crack propagation rate with time.

The basic idea of this paper is to model the process

$\underline{X}(t)$  as the output of a filter driven by a Gaussian white noise:

$$\underline{X}(t) = \mu + \rho \cos \underline{\Phi}(t) \dots \dots \dots (4a)$$

$$d\underline{\Phi}(t) = \sigma d\underline{w}(t) \dots \dots \dots (4b)$$

where  $\mu$  is the mean value of the process  $\underline{X}(t)$ ,  $\underline{w}$  denotes a Wiener process and  $\sigma$  and  $\rho$  are parameters of the model. The crack propagation rate is governed, then, by the following stochastic differential equation:

$$d\underline{a}(t) = Q(\underline{a}) [\mu + \rho \cos \underline{\Phi}(t)] dt \dots \dots \dots (5a)$$

$$d\underline{\Phi}(t) = \sigma d\underline{w}(t) \dots \dots \dots (5b)$$

Since  $-1 \leq \cos \underline{\Phi}(t) \leq 1$  and  $\mu > \rho$ , the incremental crack length  $d\underline{a}(t)$  is certainly non-negative. The solution of Eq.(5) is a Markov process with transition probability density function  $p_{\underline{a}}(a, t)$ .

SOLUTION SCHEME

With the objective of estimating the function  $p_{\underline{a}}(a, t)$  the following change of variable is introduced:

$$\underline{b}(t) = \left[ \int_{a_0}^{\underline{a}} dv/Q(v) \right] - \mu t \dots \dots \dots (6)$$

In this way Eq.(5) can be re-written as:

$$d\underline{b}(t) = \rho \cos \underline{\Phi}(t) dt \dots \dots \dots (7a)$$

$$d\underline{\Phi}(t) = \sigma d\underline{w}(t) \dots \dots \dots (7b)$$

For the joint transition probability density function  $p_{\underline{b}, \underline{\Phi}}(\underline{b}, \underline{\Phi}, t)$  the following Fokker-Planck equation holds:

$$0 = \partial p_{\underline{b}, \underline{\Phi}}(\underline{b}, \underline{\Phi}, t) / \partial t + \partial [(\rho \cos \underline{\Phi}) p_{\underline{b}, \underline{\Phi}}(\underline{b}, \underline{\Phi}, t)] / \partial \underline{b} - (1/2) \partial^2 [\sigma^2 p_{\underline{b}, \underline{\Phi}}(\underline{b}, \underline{\Phi}, t)] / \partial \underline{\Phi}^2 \dots \dots \dots (8)$$

In order to calculate an approximate solution of this partial differential equation the first step is the evaluation of the moments of the process  $\underline{b}(t)$  by means of Itô calculus (4). Itô formula must be written for Eq.(7a) and (7b) with special function  $u(\underline{b}(t), \underline{\Phi}(t)) = [\underline{b}(t)]^k \cos(n\underline{\Phi}(t))$ :

Eq. (7a) and (7b) with special function  $u(\underline{b}(t), \underline{\phi}(t)) = [b(t)^k \cos(n\phi(t))]$ :

$$d[(\underline{b}(t))^k \cos n \underline{\phi}(t)] = -n (\underline{b}(t))^k \sin n \underline{\phi}(t) \sigma d\underline{w}(t) - 1/2 \sigma^2 \cdot n^2 (\underline{b}(t))^k \cdot \cos n \underline{\phi}(t) dt + 1/2 \rho k \underline{b}(t)^{k-1} [\cos(n-1)\underline{\phi}(t) + \cos(n+1)\underline{\phi}(t)] dt \dots (9)$$

Put  $c_{k,n}(t) = E[(\underline{b}(t))^k \cdot \cos n \underline{\phi}(t)]$ . Taking the mean value of Eq.(9) ( $E[d\underline{w}(t)]=0$ ), the following system of linear differential equations can be written for moment  $c_{k,n}(t)$ :

$$\dot{c}_{k,n}(t) = - 1/2 \sigma^2 n^2 c_{k,n} + 1/2 \rho k(c_{k-1,n+1}(t) + c_{k-1,n-1}(t)) \quad k,n = 0, 1, \dots \dots (10)$$

System (10) can be solved in a numerical way. The moments of the process  $\underline{b}(t)$  are given by  $c_{k,0}(t)$ .

NUMERICAL EXAMPLE

In order to check the result of this moment calculation two different types of simulation techniques (7) were used. One of this techniques performs the simulation of the Wiener process  $\underline{w}(t)$  in the time domain. Let  $w'(t)$  a realization of this process, then Eq.(7a) becomes:

$$d\underline{b}(t) = \rho \cos(\sigma w'(t)) dt \dots \dots \dots (11)$$

The second approach operates in the frequency domain and carries out realizations of a white-noise, which is the formal derivation of a Wiener process. Then the direct integration of Eq.(7a) and (7b) is conducted. This approach offered several difficulties.

The numerical example given here is an aluminium specimen (6) with a central hole. The crack is in the range of small crack size and the value of  $Q(\underline{a}(t))$ ,  $a_0$ ,  $\mu$ ,  $\rho$  and  $\sigma$  are:

- $Q(\underline{a}(t)) = \underline{a}(t)$
- $a_0 = 0.004$  [inches]
- $\mu = 1.13 \cdot 10^{-1}$  [inches/fh x 1000]
- $\rho = 2.6 \cdot 10^{-2}$  [inches/fh x 1000]
- $\sigma = 0.6$

The values of the first moments obtained by Eq.(10) and by simulation are summarized in Table I. These moments will be used then to approximate the expression of the probability density function of the process  $\underline{a}(t)$ .

#### CONCLUSION

Starting from the knowledge of the distribution of the stress intensity factor is possible to use the method illustrated in this paper to estimate the probability density function of crack size. The probability distribution of the stress intensity factor can be evaluated by means of techniques given in Refs. (8) and (9).

Table I. Moments of order  $k$  of the random process  $\underline{b}(t)$  computed by:  
a) Itô calculus; b) simulation of realizations of the Wiener process.  
The sample size for the simulation is 100.

k	2	4	6	8	10
a)	3.95 E-02	2.93 E-03	2.65 E-04	2.66 E-05	2.84 E-06
b)	3.87 E-02	2.98 E-03	2.67 E-04	2.57 E-05	2.60 E-06

#### SYMBOLS USED

$W(T)$	= Wiener process
$P_{b, \phi}(b, \phi, t)$	= joint density of process $\underline{b}(t)$ and $\underline{\phi}(t)$
$\underline{b}(t)$	= auxiliar variable
$P_{\underline{a}}(a, t)$	= transition probability density function of crack length $\underline{a}(t)$

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