STRESS SINGULARITIES IN AXISYMMETRIC NOTCHED BODIES

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A numerical technique is presented for the accurate dynamic analysis of the stress singularity at the vicinity of a sharp angular notch in solids of revolution. The singular fields are obtained by comparison with those corresponding to plane problems. The extension of the method to crack geometries is discussed and numerical examples are also presented.

INTRODUCTION

The analysis of structures containing cracks or notches has received much attention in the theory of Elasticity. It is well known that such analysis leads to stress singularities (1,2). A very simple approach is used in this paper to determine the singular stress and displacement fields in the neighbourhood of the tip of a sharp angular notch in axisymmetric bodies. This is done by comparing locally the axisymmetric case with plane and antiplane problems. This solution is used to define a special finite element which takes into account the exact form of the singular stresses by means of a global-local formulation.

THEORETICAL ANALYSIS

The problem of a solid of revolution subjected to torsion or tensile loading is considered in this section. In the absence of body forces, the equilibrium equation for the torsion problem in cylindrical coordinates can be written as

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$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2 \frac{\sigma_{r\theta}}{r} = 0 \tag{1}$$

Let us define the coordinate system (s,φ), Fig. 1. The equilibrium in terms of the displacement u_θ has the following form:

$$\frac{\partial^2 u_\theta}{\partial s^2} + \frac{1}{s}\frac{\partial u_\theta}{\partial s} + \frac{1}{s^2}\frac{\partial^2 u_\theta}{\partial \varphi^2} + \frac{1}{a + s\cos(\alpha+\varphi)}\left[\frac{\partial u_\theta}{\partial s}\cos(\alpha+\varphi) - \frac{\partial u_\theta}{\partial \varphi}\frac{\mathrm{sen}(\alpha+\varphi)}{s}\right] - \frac{u_\theta}{[s\cos(\alpha+\varphi) + a]^2} = 0 \tag{2}$$

In order to obtain the dominant singular fields, the neighbourhood of the notch tip, s<<a, is considered. Then, equation (2) can be approximated as

$$\frac{\partial^2 u_\theta}{\partial s^2} + \frac{1}{s}\frac{\partial u_\theta}{\partial s} + \frac{1}{s^2}\frac{\partial^2 u_\theta}{\partial \varphi^2} + \frac{1}{a}\left[\frac{\partial u_\theta}{\partial s}\cos(\alpha+\varphi) - \frac{\partial u_\theta}{\partial \varphi}\frac{\mathrm{sen}(\alpha+\varphi)}{s}\right] - \frac{u_\theta}{a^2} = 0 \tag{3}$$

This equation can be compared with that corresponding to the antiplane problem, where equilibrium in polar coordinates has the form:

$$\frac{\partial^2 u_z}{\partial s^2} + \frac{1}{s}\frac{\partial u_z}{\partial s} + \frac{1}{s^2}\frac{\partial^2 u_z}{\partial \varphi^2} = 0 \quad \Longleftrightarrow \quad \nabla^2 u_z = 0 \tag{4}$$

Taking into account the expected form of the displacement: $u_\theta = s^\gamma G(\varphi)$, expressions (3) and (4) can be regarded as equivalent equations. On the other hand, the stress-displacement relations for the torsion problem in the (s,φ) coordinate system are:

$$\sigma_{st} = \mu \left[\frac{\partial u_\theta}{\partial s} - \frac{u_\theta}{s\cos(\alpha+\varphi) + a}\cos(\alpha+\varphi)\right]$$
$$\sigma_{\varphi t} = \mu \left[\frac{1}{s}\frac{\partial u_\theta}{\partial \varphi} + \frac{u_\theta}{s\cos(\alpha+\varphi) + a}\mathrm{sen}(\alpha+\varphi)\right] \tag{5}$$

Confining the analysis to the singular region, s << a, and taking into account the form of the displacement, the singular stresses become:

$$\sigma_{st} = \mu\, s^{\gamma-1}\,\gamma\, G(\varphi) \qquad \sigma_{\varphi t} = \mu\, s^{\gamma-1} G'(\varphi) \tag{6}$$

Equations (6) are the same as those corresponding to the antiplane problem (3). In addition, boundary conditions should be considered. These are identical for both problems and reduce to assume stress free notch faces. Therefore the displacement can be expressed as:

$$u_\theta = \frac{T}{\mu} s^\gamma \cos(\gamma\varphi) \qquad \gamma = \frac{n\pi}{ß} \tag{7}$$

The particular case of a crack can be readily obtained from the general formulation by putting $ß = 2\pi$, so that KIII is related to the parameter T, T = - KIII $(2/\pi)^{1/2}$

The second problem considered in this paper is that of an axisymmetric notched body subjected to tensile loading. A similar procedure to that described above allows to identify the resulting equilibrium equations, stress-strain relationships and boundary conditions referred to the local coordinates (s,φ) with those corresponding to a notch in plane elasticity (4,5). Hence the well established solutions for plane problems (6,7) can be used to define the singular fields in the axisymmetric problem. The special case of cracks in axisymmetric problems requires a different formulation in the sense that two intensification coefficients, KI and KII, are needed to define the singular fields. Details of this analysis can be found in references (4,5).

It should be noted that dynamic effects do not modify the form of the solution, because of the different orders of singularity in the inertia terms of the equation of motion. In this case the intensification coefficients have to be regarded as time dependent parameters.

FINITE ELEMENT FORMULATION

The effect of singularity is included in an eight-noded element by adding a global term in the interpolation of displacements (8).

$$u_\theta = N_j d_\theta^j + T (F_\theta - N_j \bar{F}_\theta^j) \tag{8}$$

for the torsion problem and similarly for the axisymmetric problem, where N_j are local shape functions, d_i^j are nodal displacements, T is a global degree of freedom and F_i^j is the value of F_i at node j, F_i being the relation between the singular displacement and the intensification coefficient. To maintain inter-element compatibility, transition elements have been defined. In these elements the global interpolation is corrected by suitable weight functions taking the value 1 on boundaries adjacent to special elements and the value 0 on boundaries adjacent to isoparametric elements. Using this interpolation, the intensification coefficients are calculated as extra degrees of freedom of the finite element model.

NUMERICAL RESULTS

Two different geometries have been considered. The first problem is that of a cracked cylinder as depicted in Fig. 2, with material properties ρ=7829 Kg/m^3, E=2.06x10^{11} N/m^2, ν=0.3. Loading conditions are: (i) Heaviside step function torque applied at both edges (torsion problem) and (ii) uniform tensile stress with a Heaviside step function time dependence (axisymmetric problem).

The second problem is depicted in Fig. 3 and corresponds to a notched cylinder with material properties ρ=2450 Kg/m^3, E=75.61 N/m^2, ν= 0.286 and the same loading conditions.

Torsional results: Fig. 4 shows the time variation of KIII in the cracked specimen. The solution is found to be in good agreement with that of Chen and Wang (9) which is also included for comparison. Time dependence of T in the second specimen is plotted in Fig 5.

Axisymmetric results: The results obtained for the cracked specimen are depicted in Fig. 6, where the magnitude of the dynamic effects should be noticed. Finally Fig. 7 shows the time dependence of the intensification coefficient in a 60º notch specimen.

CONCLUSIONS

The analysis described in this paper allows to calculate the stress singularity in the neighbourhood of a sharp notch for elastic axisymmetric bodies. Special finite elements have been defined in order to characterize the singularity. The obtained results make clear the magnitude of the dynamic effects when dealing with transient loads and show a good agreement with those found in the technical literature.

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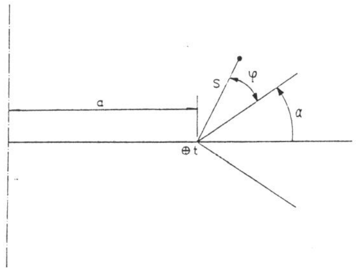

Figure 1. Local coordinates for the notch geometry

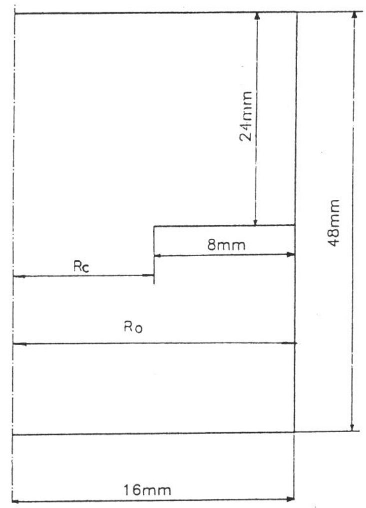

Figure 2. Cracked cylinder

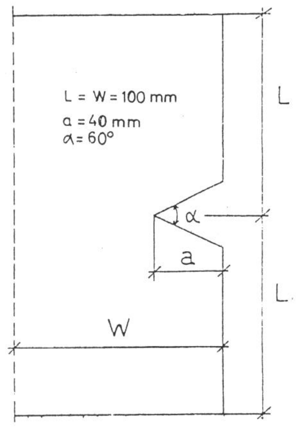

Figure 3. Notched cylinder

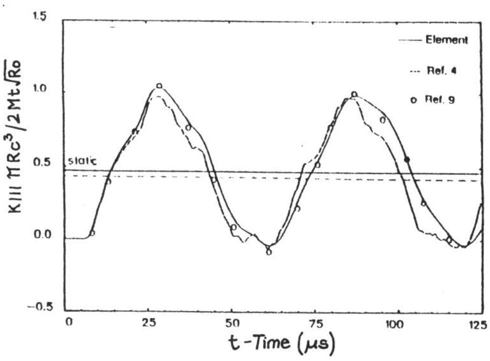

Figure 4. KIII as a function of time

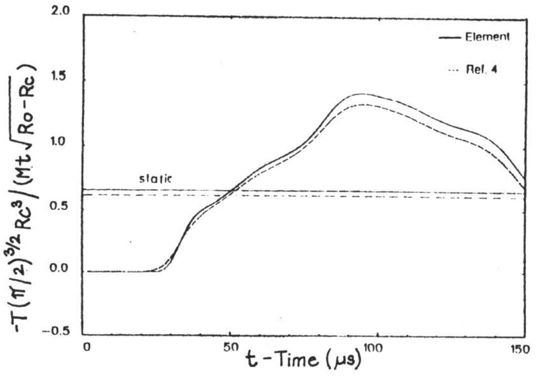

Figure 5. T as a function of time

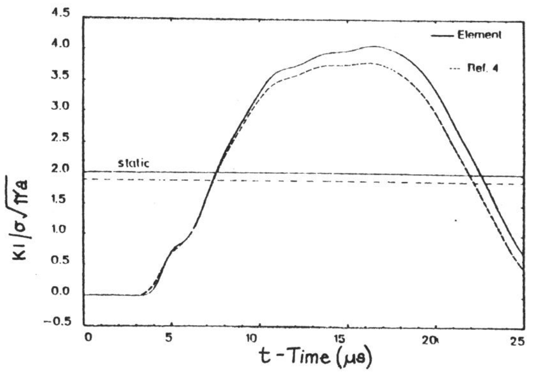

Figure 6. Variation of KI with time

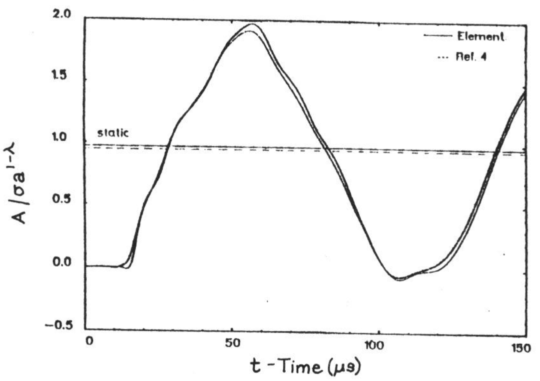

Figure 7. Variation of the intensification coefficient with time.