# THE TRANSITION FROM PLASTIC FLOW TO COMBINED FLOW & FRACTURE: THE SCALING OF $J_R$ CURVES

A.G. ATKINS*

The cracking of deeply-notched beams with uncontained plasticity is modelled in such a way that solutions for both rigid-workhardening plastic, and non-linear elastic, behaviour may be obtained and compared. The relationships between the work up to the onset of fracture, and between the work accummulated during stable propagation of the crack, and the fracture toughness (in terms of R, $J_c$ or $J_R$) are given. How $J_R - \Delta a$ data from specimens having different ligaments should condense to a single plot, using the scaled abscissa of $\Delta a/(w-a_o)$, is revealed which agrees with the results of reference [1] using a different model. The magnitudes of $J_c$ and other properties derived from such plots are different depending upon whether a rigid plastic or non-linear elastic interpretation is employed.

## INTRODUCTION

Transitions from extensive plastic flow to combined flow and fracture were considered in a paper by the author presented at the P.S. Theocaris Festschrift [1]. The line of attack was to consider competing rigid-plastic work increments for the two modes, and the load-displacement (or moment-rotation) relationships, from which the deformation at the transition may be determined. That is, comparison was made between the equation governing flow alone

$$X du \text{ or } M d\theta = d\Gamma = d(WV) = W dV + V dW \quad \text{—— (1)}$$

and that for combined flow and stable crack propagation

$$X du \text{ or } M d\theta = d\Gamma + R dA = W dV + V dW + R dA \quad \text{—— (2)}$$

*Department of Engineering University of Reading Whiteknights PO Box 225 Reading RG6 2AY U.K.

In Equations (1) and (2) X is load and u the associated displacement; M is moment and $\theta$ rotation; $\Gamma$ is irreversible plastic work, being the product of W work per volume and V volume being plastically deformed; R is the specific work of fracture in the presence of large amounts of plastic flow (the fracture toughness) and A is crack area. In setting down Equation (2) it is assumed that the variables are separable: a growing body of data not only shows that this is acceptable [2-4] but additionally demonstrates how the geometry-independent R may be uncoupled from the remote plastic flow. In [1] the case of bending of a deeply-notched beam was investigated using the model shown in Figure 1a. Expressions were derived for the work areas U beneath the $M\theta$ plot at various stages, viz: see Figure 1b,

$$U_{before} = (w-a_o)\ RB/2 \qquad (3)$$

$$U_{acc} = U_{before} + U_{after} = (w + a-2\ a_o)RB/2 \qquad (4)$$

and $U_{total} = (w-a_o)\ RB \qquad (5)$

when a = w in Equation (4).

Equation (4) may be rewritten as

$$2\ U_{acc}/B(w-a_o) = R + R\ [\Delta a/(w-a_o)] \qquad (6)$$

where $\Delta a = (a-a_o)$. This is not only reminiscent of

$$J_R = J_c + (dJ_R/da)\ \Delta a \qquad (7)$$

but, additionally, the form of Equation (6) seems to give a basis for the normalisation of $J_R$ data from different-size testpieces. It was Turner [8] who pointed out to the author the significance of Equation (6).

However, for reasons that will be discussed in detail elsewhere [10], it transpires that the notched beam solution in [1] is a <u>non-linear elastic</u> solution rather than a rigid-plastic solution. This paper now gives both a nle and rigid plastic solution for notched beams. <u>Unloading</u> of parts of the plastic volume has to be incorporated in cases of finite length ligaments and that was not properly taken account of in earlier attempts.

## RIGID -PLASTIC FRACTURE OF NOTCHED BEAM

Consider the notched beam in Figure 2a under pure bending. As it helps better to understand those parts of the beam which unload during cracking, consider flow along the whole beam length beneath the notch depth rather than being limited to flow across the ligament. With the usual assumptions, the bending strain is linear across the depth of the beam and is $\varepsilon = y/\rho$ with $\rho$ the radius of curvature, from which the non-linear rigid-plastic stress distribution is $\sigma = \sigma_o\ (y/\rho)^n$ using $\sigma = \sigma_o\ \varepsilon^n$. The associated bending moment is

$$M = 2 \int_o^{b/2} B\sigma_o\ y^{n+1}\ dy/\rho^n$$

$$= B\sigma_o\ b^{n+2}\ \theta^n/2^{n+1}\ (n+2)L^n \qquad (8)$$

where B is the thickness, b the ligament length, L is the length of the beam and $\rho\theta = L$.

When cracking starts we postulate, in this model, that the crack tip strain maintains a constant value $\varepsilon_{tip}$. The compressive side of the neutral axis unloads during propagation, and it is only the regions on the tensile side which experience increasing strain as the crack advances into them. For a rigid-plastic solid, no work is required on the parts unloading from a compressive bending strain to zero strain, and only when those parts go tensile is work consumed. Thus the $d\Gamma$ component of incremental irreversible plastic flow during propagation arises only in those tensile loading regions which comprise, in the case of a rectangular beam, one-half of the current cross-section.

Therefore, for prior plastic flow when $b = (w - a_o)$, Equations (1) and (8) give

$$Md\theta = d\Gamma = B\sigma_o\ (w-a_o)^{n+2}\ \theta^n\ d\theta/2^{n+1}\ (n+2)\ L^n \qquad (9)$$

and for combined plastic flow when $b = (w-a)$, Equations (2) and (8) give

$$Md\theta = \tfrac{1}{2}d\Gamma + RdA$$

$$= \tfrac{1}{2}B\sigma_o\ (w-a)^{n+2}\ \theta^n\ d\theta/2^{n+1}\ (n+2)^n + RBda \qquad (10)$$

At the transition from flow, to flow plus fracture, $(Md\theta_{before}) = (Md\theta_{after})$, i.e.

$$\frac{B\sigma_o\ (w-a_o)^{n+2}\ \theta_T^n}{2^{n+2}\ (n+2)\ L^n} = RB\ \frac{da}{d\theta} = \frac{RB\ (w-a_o)^2}{2L\ \varepsilon_{tip}}$$

where $\theta_T$ is the rotation at the transition, since a, $\rho$ and $\theta$ are connected during propagation by $(w-a)/2\rho = \varepsilon_{tip} = (w-a)\theta/2L$, giving $da/d\theta = (w-a)^2/2L\ \varepsilon_{tip}$.

Hence

$$\theta^{n}{}_{T,rpl} = 2^{n+2}\ (n+2)\ L^{n}R/\sigma_{o}\ (w-a_{o})^{n}\ 2L\ \varepsilon_{tip} \quad \text{\_\_\_(12i)}$$

or $$\theta^{n+1}{}_{T,rpl} = 2^{n+2}\ (n+2)\ L^{n}R/\sigma_{o}\ (w-a_{o})^{n+1} \quad \text{\_\_\_(12ii)}$$

Note that in problems with finite ligaments transitions are predicted to occur even with n = 0 in the second of Equation (12), owing to the unloading in part of the ligament. In problems with infinite ligaments such as tearing [5], $\theta_T \rightarrow \infty$ for n = 0 since continued collapse is favoured once started.

We also note that since $\theta_{T,rpl}\ (w-a_o) = 2L\ \varepsilon_{tip}$ we conclude that

$$\varepsilon_{tip}{}^{n+1} = 2(n+2)R/\sigma_o\ L \quad \text{\_\_\_ (13)}$$

from the first of Equation (12). Observe the connexion in this model between $\varepsilon_{tip}$ and L.

Using $\theta^n = (2L\varepsilon_{tip})^n/(w-a)^n$ and $(da/d\theta) = (w-a)/\theta = 2L\varepsilon_{tip}/\theta^2$ in Equation (10) we find

$$M_{after} = 2B(2L\varepsilon_{tip})R/\theta^2$$

$$= \left[2BR/\theta^2\right] \left[\frac{2^{n+2}(n+2)\ L^{n}R}{\sigma_o(w-a_o)^{n}\ \theta^{n}{}_{T,pl}}\right] \quad \text{—(14)}$$

using the first of Equation (12) for $2L\varepsilon_{tip}$. $M_{before}$ is, of course, given by Equation (8) with $b = (w-a_o)$.

We may determine the various external work components as follows:

$$U_{before} = \int_{o}^{\theta_T} M_{before}\ d\theta$$

$$= 2BR\ (w-a_o)/(n+1) \quad \text{——(15)}$$

$$U_{after} = \int_{\theta_T}^{\theta} M_{after}\ d\theta$$

$$= 2BR\ \Delta a \quad \text{——(16)}$$

where $\Delta a = (a - a_o)$, after some manipulation. Hence

$$U_{acc} = 2BR\ (w-a_o)/(n+1) + 2BR\ \Delta a \quad \text{\_\_\_(17)}$$

When $\Delta a = (w-a_o)$,

$$U_{total} = 2BR\ (w-a_o)\ (n+2)/(n+1) \qquad (18)$$

## NON-LINEAR ELASTIC SOLUTION

If the beam in Figure 2 a is assumed to behave in a non-linear elastic, rather than rigid-plastic, manner Equation (8) gives the reversible $M\theta$ relation at constant crack area $bB = (w-a)B$, and Equation (9) gives now, not $d\Gamma$ but $d\Lambda$, the increment of stored recoverable elastic strain energy. There is no remote plastic flow in this formulation. Hence

$$\Lambda = \int M d\theta = B\sigma_o\ (w-a)^{n+2}\ \theta^{(n+1)}/2^{n+1}(n+2)\ L^n(n+1) \qquad (19)$$

Application of $R = -\ (\partial\Lambda/\partial A)_\theta$ [9] then gives

$$R = \sigma_o\ \theta^{n+1}\ (w-a)^{n+1}/2^{n+1}\ (n+1)\ L^n \qquad (20)$$

or, in terms of $M\theta$ during crack propagation along a constant R locus,

$$M = 2\ L^{n/(n+1)}\ R^{(n+2)/(n+1)}\ B\ (n+1)^{n+2}/(n+2)\ \sigma_o^{1/n+1}\ \theta^2 \qquad (21)$$

For a starter crack of length $a_o$, the transition rotation $\theta_T$ at which simple elasticity transforms to propagation is given by Equation (20) with $(w-a_o)$ for the crack length, i.e.

$$\theta_{T,nle}^{n+1} = 2^{n+1}\ (n+1)\ L^n\ R/\sigma_o(w-a_o)^{n+1} \qquad (22)$$

We note that the non-linear elastic $\theta_T$ given by Equation (22) is not the same as the rigid-plastic $\theta_T$ given by Equation (12). In fact

$$\left[\theta_{T,nle}/\theta_{T,rpl}\right]^{n+1} = (n+1)/2(n+2)$$

and $\theta_{T,nle} < \theta_{T,rpl}$.

Using the same nomenclature for work areas U as for the rigid-plastic case, $U_{before} = \Lambda$ at $\theta = \theta_T$, i.e.

$$U_{before} = RB(w-a_o)/(n+2) \qquad (24)$$

Again, $U_{after} = \int_{\theta_T}^{\theta} M d\theta$ with M given by Equation (21) whence

$$U_{after} = RB(n+1)\,\Delta a/(n+2) \qquad (25)$$

It follows that

$$U_{acc} = RB\,(w-a_o)/(n+2) + RB(n+1)\Delta a/(n+2) \qquad (26)$$

## COMPARISON OF PREDICTION OF THE TWO MODELS

Figure 2b shows the $M\theta$ relations predicted by the rigid-plastic and non-linear elastic formulations. Before cracking, Equation (8) gives the same prediction for both, the difference being that all the external work is dissipated in plastic flow in the former case (unloading lines being vertically downwards), whereas in the non-linear elastic case all the external work is stored as elastic strain energy and is recoverable.

During propagation, both Equation (14) and Equation (21) predict that $M \propto 1/\theta^2$ but the level of the curves is different and the rotation $\theta_T$ at which cracking commences is different. According to Equation (23) $\theta_{T,rpl} = 2\theta_{T,nle}$ for $n = 0$ and $\theta_{T,rpl} = (10/3)^{2/3} = 2.23\theta_{T,nle}$ for $n = 0.5$ which are the extremes of n encountered in the Ludwik $\sigma = \sigma_0\varepsilon^n$ relation for most metals. The ratio of the greatest moments (i.e. at the respective $\theta_T$) is, according to Equation (8),

$$M_{rpl}/M_{nle} = [\theta_{T,rpl}/\theta_{T,nle}]^n = [2(n+2)/(n+1)]^{n/(n+1)} \qquad (27)$$

which is unity for $n = 0$, and $(10/3)^{\frac{1}{3}} = 1.49$ for $n = 0.5$. Clearly with low n, and larger $\theta_T$, the combined curves appear 'flatter' for a longer period.

Of course, the analysis is likely to be used to interpret a given experimental $M\theta$ plot obtained with a given pre-cracked beam of given material behaviour, rather than to assess the performance of two separate testpieces of "known" rigid-plastic and "known" non-linear elastic behaviour. In particular, we usually wish to determine R (or $J_c$) and $J_R$ behaviour from the measured work areas. While the rigid-plastic and non-linear elastic formulations predict relations of similar form between R or $J_C$ and $U_{before}$, and between $J_R$ and $U_{acc}$ and crack growth $\Delta a$, it is clear that the absolute values of R, $J_C$ and $J_R - \Delta a$ derived from a given experimental plot will be different.

The nle solution for the deeply-notched beam suggests that Turner's $\eta$ is (n+2); the rigid plastic solution gives $\eta$ = (n+1)/2 which is ½ for n = 0. There is thus a factor of four difference in this parameter according to the two interpretations. Again, look at the differences between plots of 2 $U_{acc}/B(w-a_o)$ versus $\Delta a/(w-a_o)$. The nle analysis predicts an intercept of 2R/(n+2) which is R for n = 0 [1], but the rigid plastic analysis gives 4R/(n+1) for the intercept which is 4R for n = 0. So, is the $J_R$ vs $\Delta a/(w-a_o)$ plot intercept equal approximately to R, or to 4R? Again, is the slope 2(n+1)R/(n+2) (= R for n = 0) as given by the nle analysis, or is it 4R as given by the rigid-plastic algebra?

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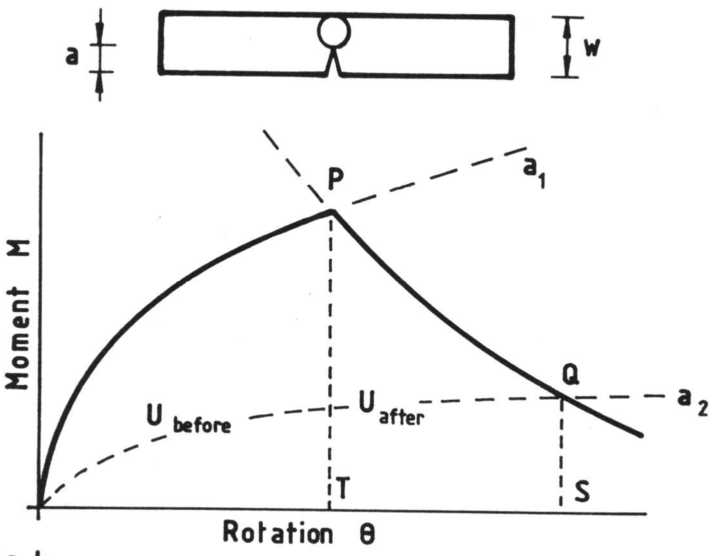


FIG. 1 a,b

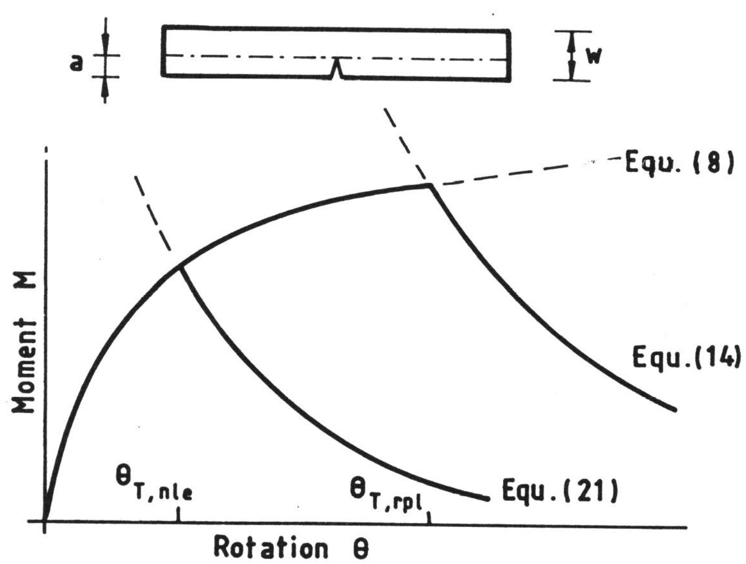


FIG. 2 a,b