ON THE USE OF THE J INTEGRAL IN FRACTURE MECHANICS

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The Gibbs free energy change due to the movement of elastic singularities in a loaded body is obtained in terms of an invariant integral from which both the Eshelby and J integral can be derived. This derivation of the J integral makes its physical significance particularly clear. It is concluded that non-linear elasticity cannot be used, in principle, to model plastic deformation and that some experimental methods used to determine the value of the J integral do not in fact measure the J integral.

Consider an elastic body having arbitrary surface tractions over its surface, Σ_o, such as to keep it in static equilibrium. Regard this body as a thermodynamic system with boundary Σ_o. For any quasistatic isothermal process and according to the first and second laws of thermodynamics the work done $(d\omega)_T$ on the system by the surface tractions (counted as negative) is

$$(d\omega)_T = -(dU - T\,dS)_T = (dF)_T \qquad (1)$$

where U is the internal energy of the body, S its entropy, F its Helmholz free energy and T thermodynamic temperature. The strain energy of a body is thus part of its Helmholtz free energy and so is its surface energy.

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If the stress tensor is σ_{ij} then the force acting on the element dS_k of Σ_o is $\sigma_{ij}\, dS_j$ and the Gibbs free energy, G, of the body is defined by

$$G = F - \int_{\Sigma_o} u_i\, \sigma_{ij}\, dS_j \tag{2}$$

where u_i is the displacement of the surface element dS_j.

If some infinitesimal change takes place in the body, at constant temperature with the surface tractions held constant, taking it from a state characterised by surface displacements u_i to a state characterised by displacements $u_i + du_i$ then the Gibbs free energy changes by

$$(dG)_{T,\sigma} = (dF)_{T,\sigma} - \int_{\Sigma_o} du_i\, \sigma_{ij}\, dS_j \tag{3}$$

$(dG)_{T,\sigma}$ will be zero if the system is in equilibrium, stable or unstable.

The infinitesimal change to which equation (3) refers might be some displacement, rotation or extension of an elastic singularity such as a crack or a dislocation loop.

As a particular example consider the extension of a zero volume crack from an arbitrary perimeter s with each element, δl, of the crack front advancing by a vector $\delta\xi_k$. In this case $(dG)_{T,\sigma}$ is the sum of the change in the elastic energy of the system and the work done by the external force exerting system plus any other energy changes such as might be due to a change, da, in surface area as the crack advances.

If γ is the surface energy (the Helmholtz free energy per unit area of surface) and dF_e is the change in strain energy then Equation (3) gives

$$(dG)_{T,\sigma} = (dF_e)_{T,\sigma} + 2\gamma\, (da)_{T,\sigma} - \int_{\Sigma_o} du_i\, \sigma_{ij}\, dS_j \tag{4}$$

Equation (4) gives the difference between the Gibbs free energy of the system with a crack with perimeter s' and the system with a crack of perimeter s for a given temperature and given surface tractions. It is possible to devise a reversible process which allows the passage between these two states. At constant temperature and constant external forces a cut is made along the crack boundary, s, extending it to its new position, s', but holding the new crack surfaces together by applying tractions to them to keep them closed. These tractions are then changed reversibly until they are zero and the enlarged crack is fully open. During this process work, $d\omega_c$, is done by the system at the new crack surfaces and $d\omega_o$ at the external surface, Σ_o. The work done at the new crack surfaces, $d\omega_c$, will have two terms, $d\omega_{cc}$ due to the cohesive forces and $d\omega_{ce}$ due to the elastic forces. According to Equation (3) the change in the Helmholtz free energy of the body is

$$(dF)_{T,\sigma} = (dF_e)_{T,\sigma} + 2\gamma\,(da)_{T,\sigma} = -\,(d\omega_{cc} + d\omega_{ce} + d\omega_o)_{T,\sigma} \qquad (5)$$

Combining Equations (4) and (5), noting that $dw_o = \int du_i\,\sigma_{ij}\,dS_j$ and $d\omega_{cc} = -\,\gamma\,(da)$, gives

$$-\,d\omega_{ce} = (dG_e)_{T,\sigma} = (dF_e)_{T,\sigma} - \int_{\Sigma_o} du_i\,\sigma_{ij}\,dS_j \qquad (6)$$

where $(dG_e)_{T,\sigma}$ is the change in Gibbs free energy excluding the change in energy due to the change in surface area of the crack.

Suppose that a new system is defined by an arbitrary surface Σ' within Σ_o but wholly enclosing the crack. The part of the body outside Σ' can now be regarded as part of the mechanical system applying tractions to Σ', and the reversible transition between states can be carried out on the new system. Since the old and new crack surfaces lie wholly within Σ', the work done by the new system at these surfaces is $d\omega_c$ $(= d\omega_{ce} + d\omega_{cc})$ and is the same as for the system defined by Σ_o. However the strain energy change within Σ' will be different from that within Σ_o, and the work done by the system on Σ' will differ from that done on Σ_o. The equivalent of Equation (6) is

$$-\,d\omega_{ce} = (dG_e)_{T,\sigma} = (dF_e')_{T,\sigma} - \int_{\Sigma'} du_i\,\sigma_{ij}\,dS_j \qquad (7)$$

where Fe' is the strain energy within Σ'. It is emphasised that the subscript s indicates that the tractions are constant on Σ_o. In general the tractions on Σ' will change during the extension of the crack.

If the change in strain energy within Σ' is expressed in terms of the change in strain energy density, W, Equation (7) then becomes

$$(dG_e)_{T,\sigma} = \int_{V'} \delta W\,dV - \int_{\Sigma'} du_i\,\sigma_{ij}\,dS_j \qquad (8)$$

where the first integration is carried out over the volume V' within Σ' and the second over the surface Σ'. The right hand side of Equation (8) is invariant and independent of the choice of Σ' provided that it encloses the crack, and is equal to the Gibbs free energy change occasioned by crack extension for the whole system defined by Σ_o. It should be noted that no assumptions have been made about linearity or small strains in deriving Equation (8).

In the derivation of Equation (8) lies the physical basis for the invariant integrals used in fracture and both the Eshelby integral [1] and the J integral [2] can be derived from it.

Instead of the extension of a crack consider the arbitrary expansion of a dislocation loop from a perimeter s to a perimeter s' its area changing by da. The resulting free energy change is again given by equation (3). Imagine that the dislocation loop is expanded by a similar quasi-static process as that used to extend the crack. This will reproduce the same equations (7) and (8) which are invariant with respect to the choice of surface Σ' provided that it wholly encloses the dislocation loop. It can be shown that the quantity $-(\partial G/\partial a)_{T,\sigma}$ is equal to the average value of the component of the crack extension force, or of the force on the dislocation, in the direction of movement [3].

If the body contains both cracks and dislocations (or any other elastic singularities) which are arbitrarily displaced Equation (8) gives the Gibbs free energy change due to these displacements and it is invariant with Σ' provided only that Σ' encloses all the moving singularities. However if Σ' encloses only some of the moving singularities Equation (8) is not invariant since it depends on the number of singularities inside Σ' and it gives the contribution to the change in Gibbs free energy of Σ_o resulting from the movement of the singularities within Σ'. If a surface Σ' is chosen which only contains an infinitesimal element, δl, of the crack front the vector crack extension force, ς_i, is given by

$$(d^2 G_e)_{T,\sigma} = - \varsigma_i \, \delta\xi_i \, \delta l \tag{9}$$

where the left hand side is a second order differential quantity and $d\xi_i$ is the vector displacement of the crack front. The generalised crack extension force will vary from point to point along the crack front. The total (first order differential) change in Gibbs free energy, as given by Equations (8) and (10), can be found from

$$(d\, G_e)_{T,\sigma} = - \int \varsigma_i \, \delta\xi_i \, \delta l \tag{10}$$

The local force per unit length on a dislocation, or on any other elastic singularity, can be defined in the same way as for the crack. If the element of crack is extended in the presence of dislocations, or other singularities, by the reversible process described above, there will be on the new crack surfaces tractions arising from the stress field of the dislocations and these will contribute to the value of the crack extension force. Hence the dislocations may be regarded as exerting a force (per unit length) on the crack and conversely, by an identical argument, the crack will exert a force on the dislocation line. Conservation of energy requires that these forces of interaction be the same in magnitude and opposite in sense. This effect is what is usually known as crack tip shielding by plastic deformation. The calculations are usually carried out in terms of the stress intensity factor, K, but they could equally well be performed in terms of the crack extension force.

Equation (8) has greater generality and physical clarity than the Eshelby integral of the energy momentum tensor. It gives the total change in the Gibbs free energy of the system when an arbitrary number of singularities are displaced in any arbitrary manner. This equation can be transformed into a single integral over the surface Σ' (the Eshelby integral) by imposing the restrictive condition that every element of every singularity within Σ' is displaced by the same vector $\delta\upsilon_i$. It is imagined that the body Σ_o is part of a larger body with singularities outside Σ_o producing the prescribed tractions. The Gibbs free energy changes are then computed in two stages. In stage I, instead of the singularities being moved by $d\upsilon_i$, the surfaces Σ_o and Σ' are displaced by $-\delta\upsilon_i$, keeping the singularities within Σ_o fixed relative to the larger body. In stage II the singularities outside Σ_o are adjusted to restore the prescribed tractions on Σ' in its new position [1,4].

The final result is:-

$$(d\, G_e)_{T,\sigma} = -\delta\upsilon_i \int_{\Sigma'} [W\delta_{ij} - \sigma_{kj}\ (\partial u_k / \partial x_i\ )]dS_j \tag{11}$$

where δ_{ij} is the Kroneker delta, $du_k/\, dx_i$ are the gradient of the displacements on Σ' and σ_{ki} he stresses on the surface element dS_j. A total force, ϕ_i, on all the singularities within Σ' can be defined as:

$$(d\, G_e)\, T, \sigma = -\ \phi_i\ \delta\upsilon_i. \tag{12}$$

and then $\phi_i = \int_{\Sigma'} [W\delta_{ij} - \sigma_{kj}\ (\partial u_k / \partial x_i\ )]dS_j$ (13)

which is the Eshelby integral.

By virtue of the invariance of Equations (7) and (8) Equation (13) is also independent of the choice of Σ'.

It should be noted that ϕ_i is the <u>total</u> force on all the singularities within Σ', hence ϕ_i could be zero even if the forces on the singularities are not zero everywhere (the case of a penny-shaped crack in the centre of a large body with a normal tensile stress).

The J integral is simply the two-dimensional version of Equation (13) for the case of a straight crack, extending through the thickness of a plate in plane strain or plane stress.

Assuming the crack to lie in the plane normal to x_2 and the crack front to be normal to x_1, then the x_1 component of ϕ_i gives the J integral in its familiar form

$$J = \phi_1 = \int_{\Gamma'} W\ dx_2 - \int_{\Gamma'} \left(\frac{\partial u_i}{\partial x_1}\right) T_i\ ds \tag{14}$$

where T_i is the traction and Γ' is a closed contour enclosing the crack.

Both Eshelby's integral and the J integral, evaluated over a surface Σ', give the total force on all the singularities inside the surface whether they have moved or not. They are also a measure of the overall rate of decrease of Gibbs free energy change of the whole body when all the singularities within Σ' are displaced by the same vector, $\delta \upsilon_i$. For a body in the form of a plate of thickness t with a through thickness crack in the plane normal to the x_2 axis J is given as

$$J = - \left(\frac{\partial G_e}{\partial a} \right)_{T, \sigma} \qquad (15)$$

where $\delta a = t \ \delta \upsilon_1$ is the change in the area, a, of the crack. Hence, because the J integral is normalized to unit thickness, it deceptively appears as a force per unit length of crack, or as a rate of Gibbs free energy decrease with respect to crack area in spite of the fact that contributions from crack, dislocations and any other singularities inside Σ' are included in the Gibbs free energy change. Equation (15) gives the true crack extension force only in the absence of dislocations (and any other singularities) or for a path, Γ', so close to the crack that it excludes all other singularities. Whether the J integral can provide a practical criterion for crack instability in the general case where the integration path encloses the crack and all other singularities is a matter for experiment to decide, although there seems to be little physical justification for it.

As noted previously, the derivation of the invariant integrals is valid for large strains and for non-linear elasticity. It is however important to distinguish between non-linear elasticity and plasticity. If there is no singularity inside Σ' the left hand sides of equations (7) and (8) are zero. The usefulness of non-linear elasticity to model plastic behaviour must therefore be extremely limited, since a plastic deformation field will always contain singularities whereas a non-linear field without plasticity never will. The value of the Eshelby integral and the J integral is entirely determined by the singularities inside Σ' and to model plasticity by non-linear elasticity is to eliminate most of these. Plasticity cannot be modelled by non-linear elasticity even if the mathematical relations between stress and strain are identical in both cases and even if the material is not unloaded. Further insight into this problem is provided by noting that the invariant integrals of Equations (8) and (11) are defined by reversible processes. There is no dissipative term in the energy term defining J and it is therefore incorrect to use the so-called "rate of stress working" in Equations (8) and (11). The strain energy density used in Equation (8) is elastic strain energy density in the body. In the elastic case all the work done on the boundaries of a region is stored as strain energy in the region, but in the plastic case only a fraction of the work done at the boundaries is stored even if the stress-strain relations on loading are perfectly matched. Thus the first integral in Equation (8) will yield a different result in each of the two cases although the second integral will be identical for both non-linear elastic-plastic solids with matching stress-strain relations on loading.

It is important to take note of this difference because erroneous conclusions can easily be reached from the improper use of the J integral in this context.

Some experimental methods developed to measure the J integral in fracture mechaincs are based on the measurement of the load deflection curves for a specimen, or series of specimens, with different crack lengths and from those determining the Gibbs free energy change as a function of crack length [5].

Differentiation is then supposed to give the value of J , but this is not so if plastic deformation takes place because, as explained, the J integral measures the Gibbs free energy change when all the singularities inside the integration contour are displaced by the same vector. Whilst the crack extension may be under some experimental control the movement of the dislocations producing the plastic deformation is not, since it depends on the local conditions. The Gibbs free energy changes measured by these experimental techniques are those given by the more general Equation (8) which is valid for arbitrary displacements of the singularities. In principle, the difference between the J integral and the quantity experimentally measured is not necessarily trivial. The J integral over a path enclosing a symmetrically loaded central crack in a plate would be zero, whilst the experiment would produce a finite value even in the purely elastic case.

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