COMPUTER SIMULATED CRACK PROPAGATION IN CONCRETE

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A new numerical method to determine crack propagation and load displacement relation for a structural element of concrete with or without steel reinforcement, loaded in plane strain or plane stress is described. A non-linear fracture mechanical model *the fictitious crack model* [1] is applied together with one of the indirect boundary element methods *the displacement discontinuity method* [2]. This technique has the following advantages: It only requires elements at the boundaries and at the cracks. Part of the unknown parameters in the set of linear equations, is simply the crack openings. The crack propagation path is easily determined by the algorithm, which is not the case when applying the finite element method, see [3].

1. PLAIN CONCRETE

In the following a non-reinforced concrete beam or another similar structure is described as a problem in plane strain or plane stress.

The Fictitious Crack Model

The fictitious crack model is a theory applicable to numerical calculation of crack propagation in a concrete structure or a structure of similar materials having a low ultimate tensile stress [1]. The theoretical ideas behind the method are briefly described as follows:

Consider a structural non-reinforced concrete structure, e.g. a beam with or without an initial crack. The concrete is assumed to be linear elastic up to the yield stress σ_y. When σ_y is reached a crack is supposed to develop perpendicularly to the maximum tensile stress $\sigma_1 = \sigma_y$. When the crack grows the tensile stress decreases from σ_y to zero according to a $\sigma - w$ relation. As shown in figure 1a this relation is assumed to be linear. Other relations can be assumed, for example a third order polynomial relation as shown in figure 1b. w is the displacement discontinuity in the crack. When w has reached the value w_u, see figure 1 no stress transfer takes place across the crack.

For $0 < w < w_u$ and $0 < \sigma < \sigma_y$ the two sides of the crack are not completely separated and the crack is said to be a fictitious crack, see figure 2.

If the beam has initial cracks the tensile stress at the crack tips are immediately very large which means that fictitious cracks develop at the crack tips from the very beginning of the loading. When a fictitious crack develops

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the displacement discontinuity at the crack need not be perpendicular to the crack, i.e. w has two components w_t and w_n, see figure 2. For simplicity it is assumed that only w_n gives rise to energy dissipation (mode 1 of fracture mechanics), i.e. the $\sigma - w$ relation is the following linear relation

$$\sigma_{nn} = \sigma_y(1 - \frac{w_n}{w_u}) \qquad \sigma_{nt} = 0$$

This approximation, $\sigma_{nt} = 0$, means that the stiffness and the ultimate load of the model are less than those of of the real structure. Since $w_t \ll w_n$ applies in general, the results obtained by using the model are expected to be close to those obtained applying a mixed mode model.

The Displacement Discontinuity Method

The unknown parameters in the set of linear equations are the displacement discontinuities at the boundary and at the cracks. Many such problems are two-dimensional problems, plane stress or plane strain, and for simplicity such two-dimensional problems are considered in the following.

In figure 3 it is shown how a closed curve Γ divides the infinite plane into two domains, an internal domain Ω and an external domain Ω^*. In Ω there is an initial crack C. In figure 3 the crack is drawn rather large, however, the crack is assumed to be small and there could be more than one crack. also a crack could extend to the boundary Γ.

As shown in figure 4 the boundary Γ and the crack C are divided into a number of rectilinear boundary elements with only one node at the midpoint of each element. The unknown displacement discontinuities are constant over each element. Other boundary elements could be used, for example a two node boundary element with a linear variation of the displacement discontinity.

The displacement discontinuity at element number j, the nodal point of which has the coordinates (x_j, y_j) in a global Cartesian coordinate system, is defined in a local coordinate system as shown in figure 5.

$$\Delta u_a^j = u_a^{j-} - u_a^{j+} \qquad \Delta u_b^j = u_b^{j-} - u_b^{j+}$$

u_a^{j-} is the limit of the displacement u_a^j for $b \to 0$ through negative values, and similarly for the other displacements.

For the elements at the fictitious crack w_t and w_n correspond to the displacement discontinuities defined above as follows, see figure 2.

$$w_t = -\Delta u_a \qquad w_n = -\Delta u_b \quad \text{for} \quad \Delta u_b < 0$$

The equations determining the unknown displacement discontinuities can be written

$$
\begin{aligned}
&\sum_{j=1}^{n} \begin{bmatrix} A_{ta}^{ij} & A_{tb}^{ij} \\ A_{na}^{ij} & A_{nb}^{ij} \end{bmatrix} \begin{bmatrix} \Delta u_a^j \\ \Delta u_b^j \end{bmatrix} = \begin{bmatrix} u_t^i \\ u_n^i \end{bmatrix} \\
&\sum_{j=1}^{n} \begin{bmatrix} B_{nna}^{ij} & B_{nnb}^{ij} \\ B_{tna}^{ij} & B_{tnb}^{ij} \end{bmatrix} \begin{bmatrix} \Delta u_a^j \\ \Delta u_b^j \end{bmatrix} = \begin{bmatrix} \sigma_{nn}^i \\ \sigma_{tn}^i \end{bmatrix} \\
&i = 1, 2, \ldots, n
\end{aligned}
\tag{1}
$$

The coefficients of influence are determined analytically, see [4] or [2]. There are $4n$ equations, but only $2n$ equations must be used, since at each node only two of the four parameters u_n, u_t, σ_{nn} and σ_{nt} are predetermined by the boundary conditions.

In order to obtain a unique determination of the displacements it is necessary to specify at least three displacement components on Γ. At an open crack the boundary conditions are $\sigma_{nn} = \sigma_{nt} = 0$. At the fictitious crack $\sigma_{nt} = 0$ and

$$\sigma_{nn} = \sigma_y\left(1 + \frac{\Delta u_b}{w_u}\right) \quad \text{for} \quad w_n = -\Delta u_b < w_u$$

$$\sigma_{nn} = 0 \qquad\qquad \text{for} \quad w_n = -\Delta u_b \geq w_u$$

When the equations are solved for the displacement discontinuities, then the displacements and stresses at a point with the global Cartesian coordinates $(x_i,\ y_i)$ in Ω or in $\Omega^\star$ are determined from the following set of equations.

$$
\begin{aligned}
&\begin{bmatrix} u_x^i \\ u_y^i \end{bmatrix} = \sum_{j=1}^{n} \begin{bmatrix} A_{xa}^{ij} & A_{xb}^{ij} \\ A_{ya}^{ij} & A_{yb}^{ij} \end{bmatrix} \begin{bmatrix} \Delta u_a^j \\ \Delta u_b^j \end{bmatrix} \\
&\begin{bmatrix} \sigma_{xx}^i \\ \sigma_{yy}^i \\ \sigma_{xy}^i \end{bmatrix} = \sum_{j=1}^{n} \begin{bmatrix} B_{xxa}^{ij} & B_{xxb}^{ij} \\ B_{yya}^{ij} & B_{yyb}^{ij} \\ B_{xya}^{ij} & B_{xyb}^{ij} \end{bmatrix} \begin{bmatrix} \Delta u_a^j \\ \Delta u_b^j \end{bmatrix}
\end{aligned}
\tag{2}
$$

Solving the Equations

At the fictitious crack the boundary conditions are satisfied by the method of trial and error:

As a first trial the following boundary conditions are applied:

$$\Delta u_b = 0 \qquad \sigma_{nt} = 0$$

If the calculations show $\sigma_{nn} \leq 0$ then the fictitious crack is closed.

If, on the other hand, $0 < \sigma_{nn}$, then as a second trial the stresses at the node are assumed to be

$$\sigma_{nn} = \sigma_y(1 + \frac{\Delta u_b}{w_u}) \qquad \sigma_{nt} = 0 \tag{3}$$

If now new calculations show $0 < -\Delta u_b < w_u$ then the fictitious crack is open, but there is still stress transfer across the crack, and σ_{nn} has the value determined by (3).

If, on the other hand, $w_u \leq -\Delta u_b$ then the node is not at the fictitious crack, but at the open crack extension with the boundary condition $\sigma_{nn} = \sigma_{nt} = 0$.

The loading is applied in steps, i.e. the following displacements and stresses on Γ are specified at each node:

$$u_t = \lambda \bar{u}_t \qquad \text{or} \qquad \sigma_{nt} = \lambda \bar{\sigma}_{nt}$$

$$u_n = \lambda \bar{u}_n \qquad \text{or} \qquad \sigma_{nn} = \lambda \bar{\sigma}_{nn}$$

$\bar{u}_t$ or $\bar{\sigma}_{nt}$ and $\bar{u}_n$ or $\bar{\sigma}_{nn}$ are a set of pre-fixed reasonably small displacements or stresses. λ is an integer giving the load step number.

It is assumed that the structure has only one crack tip. Modification of the procedure for dealing with more than one crack tip is straightforward, however, more complicated in programming.

Load Step 1

First, the crack propagation path is determined by calculating the stresses at a point on the tangent to the crack at a distance from the crack tip of half an element length. this is done by applying (2). The crack is assumed to propagate in the direction perpendicular to the maximum principal tensile stress. In this way the position of the first element at the fictitious crack is determined.

In case it is found by applying (2) that at a certain location the maximum principal tensile stress exceeds σ_y a fictitious crack is assumed to develop perpendicularly to the maximum principal tensile stress in exactly the same way as described above for the fictitious crack starting at the crack tip. In particular this will be the case if the structure has no initial cracks.

The boundary conditions at the node of the first element on the fictitious crack are satisfied by the method of trial and error as described above.

In case it is found that the node is closed, the crack is stable and the calculations proceed by load step two ($\lambda = 2$). In case the node is either at the fictitius crack or at the open crack extension a new element is added.

If, for an element at the fictitious crack, the calculations show $w_u \leq -\Delta u_b$ then the element is not at the fictitious crack any more, but at the open crack extension, and the calculation must be repeated under this new condition.

This procedure is continued until the last added new element does not open, $0 \leq \Delta u_b$. This means that the crack is stable for this load and the calculations proceed with load step two ($\lambda = 2$). In case the determinant of the equations (1) is very small this indicates that the structure is near a collapse in dynamic fracture even in displacement controlled loading.

Load Step 2
For $\lambda = 2$ the procedure from load step 1 is repeated. The following load steps are dealt with in the same way.

The Load Displacement Curve
The load displacement curve including the descending branch, e.g. for a beam, can be determined applying displacement controlled loading, i.e. the boundary conditions are specified as displacement components or as stress components, the latter being equal to zero. For each load step the total load (integrated stresses) must be calculated. If the load is a concentrated force as in a three-point bend test then the force must be calculated as integrated shear stresses over a cross-section of the beam, because a concentrated force cannot be determined by integrating stresses over a single boundary element. The end points of the boundary element are singular points with infinitely high stresses.

2. REINFORCED CONCRETE

Figure 6 shows the bottom of a reinforced concrete beam with a layer of reinforcing steel bars. a portion of the cross-sectional area corresponding to a single steel bar is considered. Hence, the problem is dealt with as a problem in plane strain.

As shown in figure 6 a crack perpendicular to the surface of the beam has developed. A necessary condition for the crack to open is that the bond between steel and concrete is destroyed over a certain length b in the figure.

The following model is applied. The structure is divided into two parts, steel and concrete, see figure 7, b is the distance between two adjacent fixed points at which the concrete and the steel have the same displacement. The fixed points are predetermined. Calculations can be made for different values of the distance b between the fixed points. Cracks can only develop at the midpoints between two fixed points. The material properties of the concrete should in fact be considered as random variables. This, however, is not considered in the following.

The Steel

At the fixed points the forces F_k act. They are, as shown in figure 11, positive in the negative direction of the x-axis. F_k acts at the x-coordinate x_k, $k = 1, 2, \ldots m$.

Figure 8 shows the assumed stress-strain curve for the steel. When AE is the elastic stiffness of one steel bar F_k can be calculated as follows, see figure 7.

$$F_1 = AE\varepsilon_{s1}$$

$$F_k = AE(\varepsilon_{sk} - \varepsilon_{s(k-1)}) \quad k = 2, 3, \ldots m - 1$$

$$F_m = AE(-\varepsilon_{s(m-1)})$$

$$e_{sk} = \frac{1}{b}(y_{k+1} - u_k) \quad k = 1, 2, \ldots m - 1$$

For $e_{sk} < \varepsilon_{sl}$: $\varepsilon_{sk} = e_{sk}$.

For $\varepsilon_{sl} \leq e_{sk} < \varepsilon_{su}$: $\varepsilon_{sk} = \varepsilon_{sl}$.

For $\varepsilon_{su} \leq e_{sk}$: $\sigma_{sk} = 0$ Collapse!

The displacements u_k are taken as a new set of unknown parameters. From the above equations the forces F_k are determined as functions of the displacements u_k.

The Concrete

The concrete structure is dealt with as described in part I of this paper. however, in the equations (1) the effect from the forces F_k written as function of the new variables u_k shall be added and the system of equations shall be extended by m new equations. These equations express u_k as linear functions of the displacement discontinuities Δu_a^j and Δu_b^j and the forces F_k which are functions of u_k themselves.

To develop this system of equations in Δu_a^j, Δu_b^j and u_k Kelvin's solution for plane strain is applied. For details, see [4].

Solving the Equations

The method of trial and error is applied to find if the stress in the steel is below or equal to the yield stress. The procedure of adding new elements at a fictitious crack, etc. is the same as described in part I for plain concrete. For further details, see [4].

REFERENCES

[1] Hillerborg, A.: *A Model of Fracture Analysis.* Division of Building Materials. The Lund Institute of Technology Report TVBM-3005, 1978.

[2] Crouch, S. L. and A. M. Starfield: *Boundary Element Methods in Solid Mechanics.* George Allen & Unwin, 1983.

[3] Petersson, Per-Erik: *Crack Growth and Development of Fracture Zones in Plain Concrete and Similar Materials.* Division of Building Materials. The Lund Institute of Technology Report TVBM-1006, 1981.

[4] Harder, N. A.: *Computer Simulated Crack Propagation in Concrete Beams by Means of the Fictitious Crack Method and the Boundary Element Method.* Department of Building Technology and Structural Engineering, University of Aalborg, Denmark, ISSN 0902-7513 R8822.

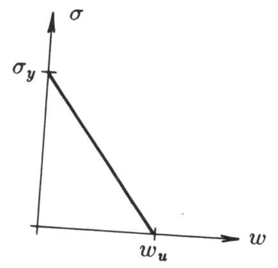

Figure 1a. Linear $\sigma - w$ relation.

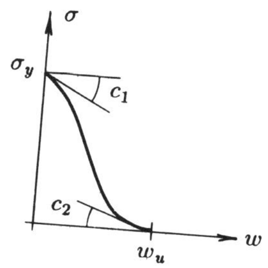

Figure 1b. Third order polynomium $\sigma - w$ relation.

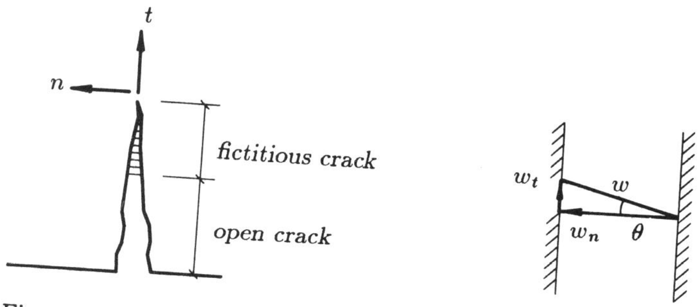

Figure 2. The fictitious crack.

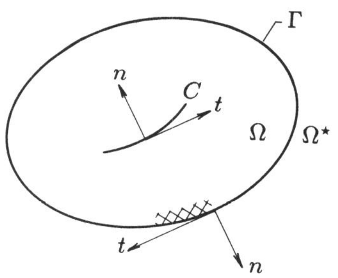

Figure 3. Boundary curve in the infinite plane.

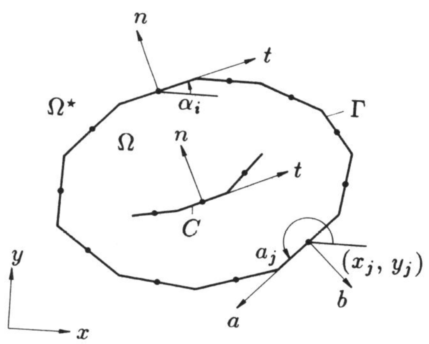

Figure 4. Boundary elements.

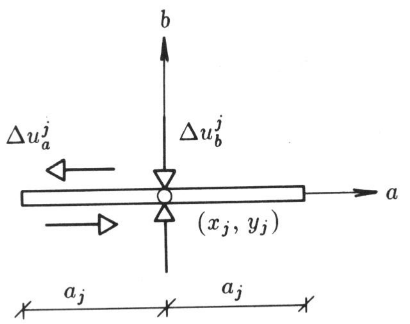

Figure 5. Displacement discontinuity.

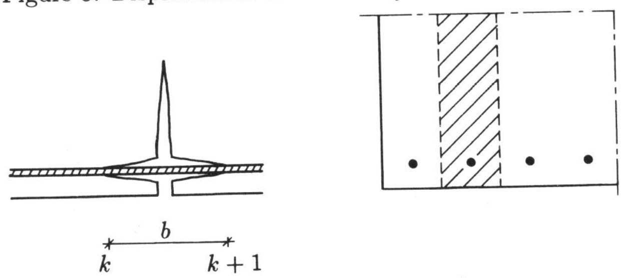

Figure 6. Crack in a reinforced concrete beam.

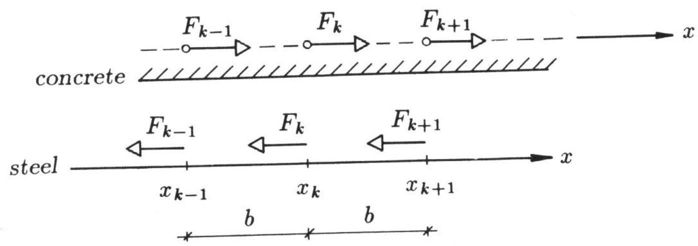

Figure 7. forces at the fixed points.

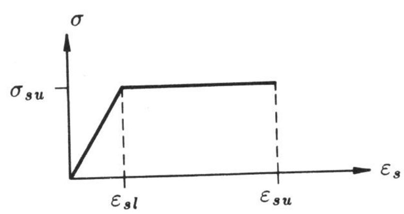

Figure 8. Stress-strain curve for the steel.