

## ANALYTICAL MODEL FOR COMPLETE MOMENT-ROTATION CURVES OF CONCRETE BEAMS IN BENDING

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When a concrete beam is subjected to rotation controlled static loading a linear cohesive crack is assumed to develop in the tensile side of the midsection of the beam. As a first approximation the two parts of the beam beyond the cohesive crack zone are assumed to perform rigid-body displacements. An elastic layer is introduced between the crack and the rigid parts of the beam. The stress distribution in the layer is assumed to be local and to be a function of the absolute displacement between the two rigid bodies. An improved model accounts for the elastic deformation of the beam via a Timoshenko beam. Different, geometrically similar, beams with constant constitutive parameters have been tested and it is shown that the moment rotation curves from the analytical model only differ slightly from the ones determined through a finite element analysis.

INTRODUCTION

Few researchers have used analytical methods based on the fictitious crack model (FC-model) to describe crack growth, and studies have been limited to very simple cases, i.e. a crack in an infinite plate, Reinhardt [3], or a limit situation of complete fracture in a beam subjected to three-point bending, Carpinteri [2].

RIGID-BODY DISPLACEMENTS

Consider an initially uncracked concrete beam with length  $L$ , depth  $h$  and thickness  $t$  subjected to a controlled rotation  $\theta^R = 2\delta/L$ , where  $\delta$  is the displacement of the midsection. The superscript  $R$  indicates rigid-body rotation. A fictitious crack is assumed to develop in the midsection, and as a first approximation the two parts of the beam beyond the fictitious crack are assumed to perform rigid-body displacements. An elastic layer with the thickness  $h^*$  is introduced around the midsection, and the stress distribution in the midsection of the beam is assumed to be local and a function of the absolute displacement between the two rigid parts.

The calculations are divided into three phases:

- A. Before the tensile strength,  $\sigma_u$  is reached in the tensile side of the beam.

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B. The evolution of the fictitious crack.

C. The actual crack propagation.

The compressive strength is assumed to be much larger than the tensile strength  $\sigma_u$ .

**Phase A.** In phase A the following constitutive relation is used for all parts of the elastic layer (superscript  $e$  indicates elastic stresses):

$$\sigma_n^e(y, \theta^R) = gw(y, \theta^R) \quad (1)$$

where  $\sigma_n^e(y, \theta^R)$  is the stress in the elastic layer,  $w(y, \theta^R)$  is the absolute distance between the two rigid parts and  $g$  is the stiffness per thickness of the elastic layer.

The neutral axis is at  $y = b(\theta^R)$  and is obtained through the equilibrium conditions, which yields  $b = h/2$ . The equivalent moment  $M$  becomes:

$$M = \frac{1}{6} \theta^R g h^3 t \quad (2)$$

In the limit situation of state A:  $w(0) = w_u$  where  $w_u$  is the absolute displacement which will induce the ultimate tensile stress, one obtains in dimensionless form:

$$M' = \frac{M}{h^2 t \sigma_u} = \frac{1}{6} \quad (3)$$

Thus, the moment-rotation curve in phase A is a straight line.

**Phase B** In phase B another constitutive relation is needed, the FC-Model. Here a linear approximation is used:

$$\sigma_n^{FC} = \sigma_u - \frac{\sigma_u}{w_c} w_{FC}(y) \quad (4)$$

where  $w_{FC}(y)$  is the crack opening displacement (COD) and  $w_c$  is the critical COD, which corresponds to no stress transmission. A situation as sketched in figure 1a will occur in phase B. The size of the fictitious crack is  $a$  and the length of the elastic tensile zone is  $b$ . The length  $b$  is determined through geometrically similar triangles. For  $y \in [0, a]$  the stresses in the fictitious crack are equal to the stresses in the elastic layer, from which COD is obtained. In order to determine  $a$  it is necessary to consider the equilibrium equations. After some manipulations the size of the fictitious crack in dimensionless form is obtained :

$$\frac{a(\theta^R)}{h} = \frac{-2 \pm \sqrt{4 + 4\alpha \left[ \left(1 - \frac{1}{\alpha}\right) \frac{w_c}{\theta^R h} - 1 \right]}}{-2\alpha} \quad (5)$$

where  $\alpha$  is a material and size dependent parameter given by:

$$\alpha = \frac{w_c}{w_c - w_u} \quad (6)$$

The equivalent moment in dimensionless form becomes:

$$M' = \frac{2\theta^R h}{w_c} \frac{\alpha}{\alpha - 1} \left[ -\frac{1}{6}\alpha \left(\frac{a}{h}\right)^3 + \frac{1}{2}\frac{a}{h} - \frac{1}{3} \right] + \frac{1}{2} \quad (7)$$

In order to stay in phase B COD for  $y = 0$  has to be less than  $w_c$ .

**Phase C.** In phase C the real crack starts to grow. The length of the real crack is  $d(\theta^R)$  and the constitutive relation is (1) and (4). For simplicity the  $y$ -axis starts where the fictitious crack ends. A situation as shown in figure 1a will occur in phase C. The length of the elastic tensile zone is obtained through the condition  $w(a) = w_u$  and the size of the fictitious crack is obtained through the condition  $w(0) = w_c$ . The length of the real crack is determined through the equilibrium conditions. Following a procedure similar to the one in phase B the length of the real crack becomes:

$$\frac{d}{h} = 1 - \left(\frac{a+b}{h}\right) \pm \sqrt{\left(\frac{a}{h}\right)^2 \alpha(\alpha - 1)} \quad (8)$$

and the equivalent moment in dimensionless form becomes:

$$M' = \frac{2\theta^R h}{w_c} \frac{\alpha}{\alpha - 1} \left[ \frac{1}{2} \frac{w_c}{2\theta^R h} \left(1 - \frac{d}{h}\right)^2 - \frac{1}{3} \left(1 - \frac{d}{h}\right)^3 - \frac{1}{6} \frac{1}{\alpha^2} \left(\frac{w_c}{2\theta^R h}\right)^3 \right] \quad (9)$$

By introducing the parameter  $\phi = 2\theta h/w_c$  it is possible to obtain complete  $M' - \phi$  curves for given  $\alpha$ .

### ELASTIC DEFORMATIONS

In this section an improved model which takes the elastic deformation into consideration will be considered. The elastic deformation,  $\delta^E$  is determined through a Timoshenko beam, Timoshenko [4] with length  $L^* = L - h^*$  and Poissons ratio  $\nu = 0.3$ :

$$\delta^E = \frac{PL^{*3}}{48EI} \left[ 1 + 2.85 \frac{h^2}{L^{*2}} - 0.84 \frac{h^3}{L^{*3}} \right] \quad (10)$$

which in dimensionless form becomes:

$$\theta^E = 2M' \epsilon_u \frac{L^{*2}}{L^2} \lambda^* \left[ 1 + 2.85 \frac{1}{\lambda^{*2}} - 0.84 \frac{1}{\lambda^{*3}} \right] \quad (11)$$

where  $\theta^E = 2\delta^E/L^*$  is the elastic rotation,  $\epsilon_u$  is the ultimate strain and  $\lambda^* = L^*/h$  is the slenderness of the beam. The total rotation is given by:

$$\theta = \theta^R + \theta^E \quad (12)$$

Before the tensile strength is reached the total rotation equals the elastic rotation of a beam with length  $L$  and slenderness  $\lambda = L/h$ . Applying this condition the elastic rotation in terms of  $L$  and  $\lambda$  becomes:

$$\theta^E = 2M' \epsilon_u \left( \lambda \left[ 1 + 2.85 \frac{1}{\lambda^2} - 0.84 \frac{1}{\lambda^3} \right] - 3 \frac{w_c}{h \epsilon_u} \left(1 - \frac{1}{\alpha}\right) \right) \quad (13)$$

Depth, $h$ (mm)	80
Width, $t$ (mm)	40
Length, $L$ (mm)	400
Modulus of Elasticity, $E$ ( $\frac{N}{mm^2}$ )	32,550
Critical Crack Opening Displacement, $w_c$ ( $\mu m$ )	76.64
Tensile Strength, $\sigma_u$ ( $\frac{N}{mm^2}$ )	2.86

Table 1: Geometry and material properties for the reference beam.

Thus, for given  $\epsilon_u$ ,  $w_c/h$ ,  $\lambda$  and  $\alpha$  it is possible to obtain the complete moment-rotation relation. It is also possible to obtain a relation between the size of the fictitious crack and the rotation. The ratio  $w_c/h$  equals  $2S_E$  where  $S_E$  is the Carpinteri brittleness number.

### RESULTS

The factor  $\alpha$  is estimated through a numerical method called the direct substructure method [1]. The estimation is made so that the maximum dimensionless moment in the analytical moment equals the maximum moment determined through the direct substructure method.

One particular beam has been considered, but the size of the beam has been varied in order to investigate size effects. The geometry of the reference beam and the material properties are listed in table 1.

In figure 2 the moment-rotation curves for the different beam sizes are plotted. It is seen on the linear part of the curves, that the analytical beam is stiffer than the beam modelled by the finite element code. This might be expected since the analytical model is based on a model which introduces more constraints on the displacements field than the FEM model. However, this discrepancy gives rise to an overall error. Despite this error the shape of the curves from the analytical solutions is almost identical to the ones obtained from the direct substructure method. Furthermore, results have shown that the ratio  $h^*/h$  does not depend on the beam size.

### CONCLUSIONS

A simple analytical model for moment rotation curves of concrete beams has been developed. The model is based on the fictitious crack model and that the crack extension path is known beforehand. The model has to be calibrated by finite element calculations in order to determine a single calibration factor. The results in this paper show that this calibration constant as a good approximation does not depend on upon the size of the beam. As an extra result the size of the fictitious crack is obtained. The model is based on a linear  $\sigma - w$  relation but is easily extended to other constitutive relations whereby the model is applicable to determine the  $\sigma - w$  relation, when three-point bending experiments are performed. The model is also applicable to notched

and reinforced concrete beams, and is therefore applicable to determine the minimum amount of reinforcement in concrete beams.

#### SYMBOLS USED

$L, h, t$  beam length, depth and thickness.

$M'$  dimensionless moment.

$\theta$  rotation.

$\delta$  displacement of midsection.

$h^*$  thickness of elastic layer.

$\sigma_n$  normal stress.

$w$  distance between rigid parts.

$g$  stiffness per unit thickness of elastic layer.

$\alpha$  material and size dependent layer.

$a$  size of fictitious crack.

$b$  length of tensile zone.

$d$  real crack length.

#### REFERENCES

#### References

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