# THE FLEXURAL STRENGTH FUNCTION FOR CONCRETE BEAMS WITHOUT INITIAL CRACKS

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The fictitious crack model is applied to develop an analytical flexural strength function for an un-notched concrete beam with rectangular cross-section. It is assumed that plane sections in the ligament remain plane and that the crack edges remain straight. A relation between the ultimate bending moment $M_u$ and the brittleness ratio $d/l_{ch}$ is given. The values of flexural strength obtained from the present study are in excellent agreements with FEM analyses for various brittleness ratios $d/l_{ch}$.

## INTRODUCTION

Since Hillerborg and his colleagues [1] introduced the fictitious crack model (FCM) to represent the fracture zone in concrete, many concrete fracture specimens and concrete structures have been analyzed successfully by this model, see, e.g. Gustafsson [2]. On the other hand, the fracture analysis based on the FCM requires the use of a finite element code which might be too cumbersome for some simple situations, e.g., to find the ultimate load carrying capacity of concrete beams. An analytical solution would make hand calculation possible and allow parametric studies such as for example the size effect in a straightforward manner. It is, therefore, of interest to develop approximate analytical solutions based on some simplifying assumptions.

If the developing path of the fracture zone is assumed in advance, according to FCM, a sharp crack with distributed wedge forces (i.e., cohesive stresses) can be arranged along this path. Take this new configuration as a primary structure. The COD influence functions of wedge forces and external loads for the primary structure can be constructed. Then, the problem can be solved by the superposition method, to satisfy the boundary conditions of the original structure. This is the basic idea behind the method presented below.

## SOLUTION BY FORCE METHOD

The tensile fracture analysis by the FCM is based on the following assumptions [1-2]:

(1) The fracture zone begins to develop when the maximum principal stress reaches the tensile strength, $f_t$ (Fig 1a).

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(2) The material in the fracture zone is partially damaged but still able to transfer stress. Such a stress is dependent on opening of the zone or "fictitious crack," $w$ (Fig 1b-c).

Because of the difficulties in satisfying the boundary conditions exactly for finite bodies, solutions are usually obtained with FEM. A closed-form solution for the fracture moment of a beam based on the FCM is presented below. We will not differentiate between three-point bending (TPB) and pure bending. Two simplifying assumptions are used:

(3) The plane section in the ligament remains plane.
(4) The fictitious crack edges remain straight.

If the strain-softening within the fracture zone is linear and the material outside that zone is linear elastic, then, a bilinear distribution of normal stresses in the cross-section where a fracture zone has a certain length $a_f$ is obtained (see Fig 2):

$$\sigma_1(y) = f_t \frac{y}{a_f} + \sigma_s \left(1 - \frac{y}{a_f}\right), \quad \text{for } 0 \le y \le a_f, \tag{1a}$$

$$\sigma_2(y) = f_t \frac{d - y}{d - a_f} - \sigma_c \frac{y - a_f}{d - a_f}, \quad \text{for } a_f \le y \le d, \tag{1b}$$

where $\sigma_s$ and $\sigma_c$ are the outermost fibre stresses in tension and compression respectively, $f_t$ is the tensile strength attained at the fictitious crack tip, refer to Fig 2. These expressions, of course, hold only for $0 \le w_s \le w_c$ (see Fig 1b).

The normal stresses, $\sigma_1(y)$ and $\sigma_2(y)$ must satisfy two equilibrium conditions: the sum of the normal forces is equal to zero and the moment of the normal forces is equal to the external bending moment $M$ ($M = Pl/4$ for a point load at mid-span). Therefore, for a rectangular cross-section of width $b$ and depth $d$, we have

$$\int_0^{a_f} \sigma_1(y)\,dy + \int_{a_f}^{d} \sigma_2(y)\,dy = 0, \text{ and } \int_0^{a_f} \sigma_1(y)\,y\,dy + \int_{a_f}^{d} \sigma_2(y)\,y\,dy = \frac{M}{b}. \tag{2}$$

$$\text{It follows that: } M = \frac{f_t\,b\,d^2}{6}\left(1 + 2\frac{a_f\,\sigma_s}{d\,f_t}\right) \text{ or } q = 1 + 2rs \tag{3}$$

where we introduced notation $q = M/(f_t b d^2/6)$, $r = a_f/d$ and $s = \sigma_s/f_t$. The COD at the crack mouth (point $S$ in Fig 2) can be expressed as follows:

$$w_s = C_{sM} M + \int_0^{a_f} C_{s\sigma}(y)\, b\, \sigma_1(y)\,dy, \tag{4}$$

where $C_{sM}$ and $C_{s\sigma}(y)$ are the influence functions of external bending moment $M$ and distributed wedges forces $\sigma_1(y)$ respectively. The following two empirical formulas developed by the author are used in this study: (As space is limited, the derivation is not given here.)

$$C_{sM} = \frac{6\,a_f}{E\,b\,(d-a_f)^2}, \quad \text{and} \quad C_{s\sigma}(y) = -\frac{6\,a_f}{E\,b\,(d-a_f)^2}(a_f - y). \tag{5}$$

Substituting in Eq (4) and using notation $r$ and $s$, we obtain

$$w_s = \frac{6\,a_f}{E\,b\,(d-a_f)^2}\left(M - \int_0^{a_f}(a_f - y)\,b\,\sigma_1(y)\,dy\right) = \frac{r(1+r+2\,r\,s)}{1-r}\frac{f_t\,d}{E} \tag{6}$$

The constitutive equation that relates stress $\sigma_s$ to displacement $w_s$, according to Fig 1b, is $\sigma_s = f_t\,(1 - w_s / w_c)$, or

$$w_s = \frac{f_t - \sigma_s}{K_{sl}}, \quad \text{for } 0 \le w_s \le w_c, \tag{7}$$

where $K_{sl} = f_t/w_c$, is the downward slope of the linear $\sigma$-$w$ curve. Eqs (6) and (7) for the COD at point $S$ can be equated (compatibility condition), leading to:

$$\frac{r(1+r+2\,r\,s)}{1-r}\frac{f_t\,d}{E} = \frac{f_t - \sigma_s}{K_{sl}}. \tag{8}$$

Solving Eq (8) for $s$, we obtain

$$s = \frac{\sigma_s}{f_t} = \frac{1-(B+1)\,r - B\,r^2}{1 - r + 2\,B\,r^2}, \tag{9}$$

in which we used notation $B = K_{sl}\,d\,/\,E$. When $w_s = w_c$, $\sigma_s = 0$. Thus, the upper bound of the argument $r$ can be found by equating Eq (9) to zero and solve for $r$:

$$r_m = \frac{\sqrt{B^2 + 6\,B + 1} - (B+1)}{2\,B}. \tag{10}$$

From Eqs (3) and (9) we obtain,

$$q = \frac{M}{f_t\,b\,d^2/6} = 1 + 2\,r\,s = \frac{1 + r - 2\,r^2 - 2\,B\,r^3}{1 - r + 2\,B\,r^2}, \quad \text{for } 0 \le r \le r_m. \tag{11}$$

When the fracture zone forms ($r = 0$), $q = 1$ and also when the fracture zone has fully developed ($r = r_m$), $s = 0$, $q = 1$. Thus, it is obvious that $q$ reaches a maximum $q_{max}$ when the fracture zone has only partially developed ($r = r_c$). The critical $r$ is found from $\partial q/\partial r = 0$, and so that

$$2\,B^2 r^4 - 2\,B\,r^3 + (4\,B - 1)\,r^2 + (2\,B + 2)\,r - 1 = 0. \tag{12}$$

One real root of this equation is $r_c$. Since the coefficients in Eq (12) are non-numerical parameters, the final form of $r_c$ is extremely complicated. An approximate value for $r_c$ is obtained as follows:

$$\bar{r}_c = \frac{1}{2}\left(\frac{1}{1.3 + 4.3\,B} + \frac{1}{\sqrt{1 + 20\,B}}\right). \tag{13}$$

It is worthwhile to note that $\Delta q \approx (\partial q/\partial r)\cdot\Delta r$, and $\partial q/\partial r = 0$ at $r = r_c$, so that the error in $q_{max}(\bar{r}_c)$ is much less than the error in using the approximate value $\bar{r}_c$ for $r_c$, and numerical tests confirmed this. Thus

$$q_{max} = \frac{M_u}{f_t\, b\, d^2/6} = \frac{1 + r_c - 2r_c^2 - 2B\, r_c^3}{1 - r_c + 2B\, r_c^2} = f(B), \qquad (14)$$

in which, $r_c$ may be calculated according to Eq (13). The quantity $M_u/(bd^2/6)$ is often denoted by $f_f$. Eq (14) represents the flexural strength function for an un-notched concrete beam with rectangular cross-section: $q_{max} = f_f / f_t = f(B)$. Noticing that $B = K_{sl}d/E$, for a given material, the flexural strength depends on the size, $d$. There are two extreme cases: When $B \to 0$, Eq (11) becomes

$$q = \frac{1 + r - 2r^2}{1 - r} = 1 + 2r, \quad \text{for } 0 \le r \le r_m, \qquad (15)$$

Here, function (15) does not have a point of maximum, but does have the greatest value $q_u = 3$ for $r_m = 1$. This agrees with the result from plastic theory. When $B \to \infty$, then $r_m \to 0$ and Eq (11) becomes $q \equiv 1$, the same as predicted by the linear elastic brittle theory. Therefore, the dimensionless number $B$ is able to describe the brittleness of the concrete beam.

## COMPARISONS WITH FEM RESULTS

To evaluate $M_u$ according to Eq (14) we need to know the value of $B$. For the linear $\sigma$-$w$ curve, $B = K_{sl}d/E$. Many results from FEM calculation reported in the literature are based on Petersson´s bilinear $\sigma$-$w$ curve (see [2] and Fig 1c). If the stress in the outermost tensile fibre $\sigma_s > 1/3\, f_t$ when $q$ reaches $q_{max}$, then only the initial slope of the bilinear curve is decisive in the load range $\{0 - M_u\}$. In these cases, it is sufficient to use the initial slope, $K_{p1}$ (Fig 1c) instead of $K_{sl}$ to calculate $B$. This conclusion has been confirmed by calculations for various values of $B$ in a practical range. Evaluating $K_{sl}$ and $K_{p1}$ in terms of the fracture energy, $G_F$, which is the area under respective $\sigma$-$w$ curve, we obtain the following formulas:

$$K_{sl} = \frac{1}{2}\frac{f_t^2}{G_F} = \frac{1}{2}\frac{E}{l_{ch}} \quad \text{and} \quad K_{p1} = \frac{5}{6}\frac{f_t^2}{G_F} = \frac{5}{6}\frac{E}{l_{ch}} \qquad (16)$$

where $l_{ch}$ is referred to as the characteristic length of material and defined by $l_{ch} = EG_F/f_t^2$. Thus we have $B = (1/2)(d/l_{ch})$ for linear $\sigma$-$w$ curve; and $B = (5/6)(d/l_{ch})$ for bilinear $\sigma$-$w$ curve.

Using Eq (11) to simulate the loading process, the controlling parameter is the relative magnitude of fictitious crack depth $r$. Following the development of a fracture zone, the maximum relative load $q_{max}$ is found. (Notice, $r \in s(r) \ge 0$ for linear $\sigma$-$w$; $r \in s(r) \ge 1/3$ for bilinear $\sigma$-$w$.) The $q_{max}$ vs $B$ diagram shown in Fig 3 is obtained in this way. (The load point displacement is not considered in the present study.)

To verify the validity of the approximate solution obtained from the present study, comparisons have been carried out with our own and other researchers´ FEM results. As an example, if linear $\sigma$-$w$ curve is used, with $f_t$ = 3 MPa, $E$ = 30 GPa, $\nu$ = 0.18, $G_F$ = 75 N/m, $d$ = 0.2 m and length $l$ = 0.8 m, our FEM calculation for TPB gives $q_{max} = (P_u l/4)/(f_t bd^2/6) = 1.44$. With these data, $q_{max}$ = 1.43 is obtained either by finding a maximum of Eq (11) numerically or by substituting $\tilde{r}_c$ in Eq (14) directly. Gustafsson [2] calculates the flexural strengths at values of the ratio $d/l_{ch}$ ranging from 0.025 to 6.4, based on Petersson´s bilinear $\sigma$-$w$ curve assumption. Such FEM results, together with the results from present formulas, are summarized in Table 1.

TABLE 1 – Comparison of results from different methods.

| $d/l_{ch}$ | $B=(5/6)(d/l_{ch})$ bilinear $\sigma$-$w$ | $q_{max}$(Zhu) Numerically | $q_{max}$(Zhu) $\tilde{r}_c$-using | $q_{max}$(Gustafsson) FEM [2] |
|---|---|---|---|---|
| 0.025 | 0.021 | 2.221 | 2.215 | 2.257 |
| 0.05 | 0.042 | 2.034 | 2.025 | 2.074 |
| 0.1 | 0.083 | 1.837 | 1.833 | 1.877 |
| 0.2 | 0.167 | 1.646 | 1.645 | 1.675 |
| 0.4 | 0.333 | 1.473 | 1.473 | 1.485 |
| 0.8 | 0.667 | 1.327 | 1.325 | 1.322 |
| 1.6 | 1.333 | 1.213 | 1.211 | 1.201 |
| 3.2 | 2.667 | 1.131 | 1.129 | 1.129 |
| 6.4 | 5.333 | 1.076 | 1.076 | 1.088 |

## CONCLUSION

A formula for the flexural strength of concrete based on the FCM is developed. The results (see Fig 3) show that the transition from ductile to brittle behaviour is governed by the brittleness number $B$, which is a function of material properties and beam depth. It can be also concluded that the ultimate bending moment of an un-notched beam is unaffected by the last part of the bilinear $\sigma$-$w$ curve proposed by Petersson. The flexural strength function found in the present study can be applied for both linear and bilinear $\sigma$-$w$ curves.

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## REFERENCES

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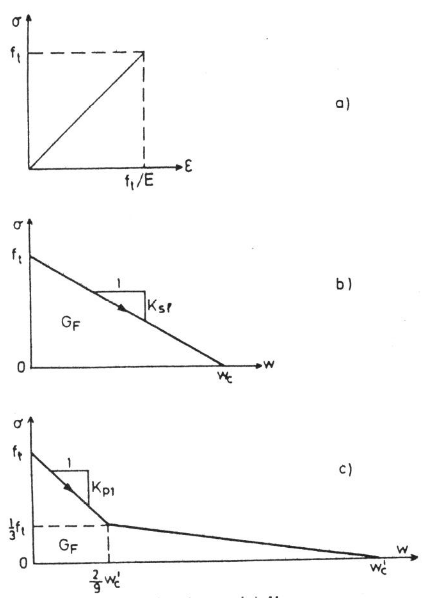


Fig 1. Constitutive laws: (a) linear $\sigma-\epsilon$, (b) linear $\sigma-w$, (c) bilinear $\sigma-w$.

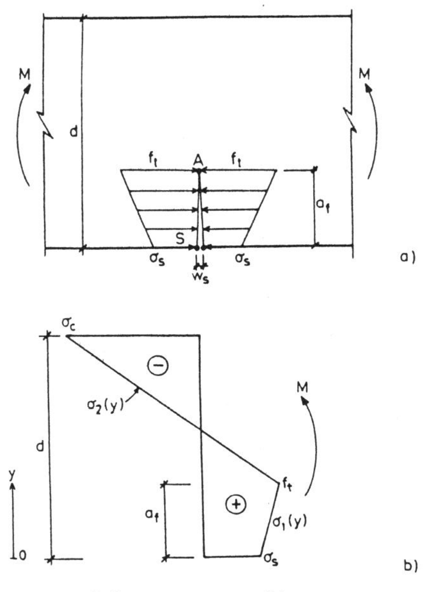


Fig 2. (a) Primary structure, (b) stress distribution in the critical section.

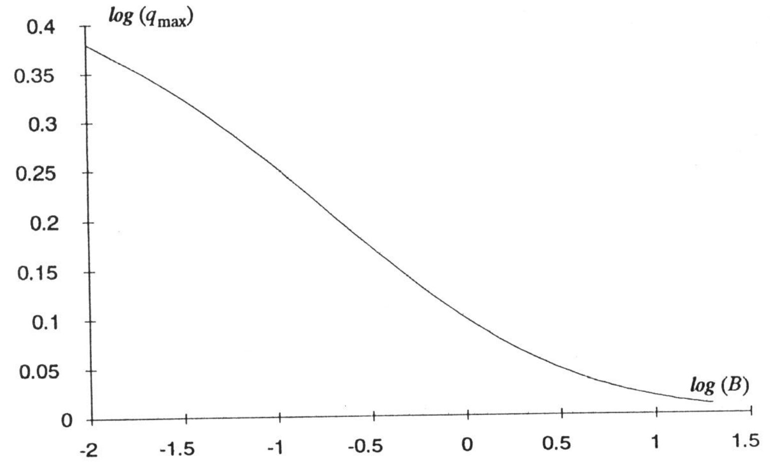


Fig 3. Relation between $log\ (q_{max})$ and $log\ (B)$. Note: $q_{max} = M_u/(f_t bd^2/6)$, $B = \lambda\ (d/l_{ch})$, where $\lambda = 1/2$ for linear $\sigma-w$, and $\lambda = 5/6$ for bilinear $\sigma-w$.