# RHEOLOGICAL MODEL FOR THE STRESS-CRACK-WIDTH RELATION OF CONCRETE UNDER MONOTONIC AND CYCLIC TENSION

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The given material model allows a realistic description of the complex behavior of concrete loaded by tension. The model consists of single rheological elements. The envelope and the hysteresis loops are described by one and the same model.

## INTRODUCTION

The post failure behavior of tensile loaded concrete is governed by the stress-crack-width relation (*$\sigma$-$w$ relation*). Up to now only a few formulations for the *$\sigma$-$w$ relation* including the cyclic behavior are published. For that reason a new material model is developed (See also (2)). This model consists of rheological elements (springs and friction blocks).

## FUNDAMENTALS

Transfer of stresses over a crack is possible due to the friction forces acting between grains and matrix. Considering every single grain being a friction block, a concrete cross section in which a crack is developing can be described with a parallel arrangement of many friction blocks. It is assumed that each friction block $i$ is working as long as the crack width $w$ is smaller than its ultimate crack width $w_{i,max}$ of this friction block.

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With $\Phi(w)$ = distribution function of ultimate crack widths, this leads to:

$$n = n_0 \cdot \Phi(w)$$

where $n$ = number of working friction blocks
$n_0$ = number of working friction blocks for zero crack width

The considered concrete area $A$ is able to transfer for zero crack width the force $P(w = 0) = f_{ct} \cdot A$. For a crack width $w > 0$ the transferable load $P(w)$ is proportional to the number of friction blocks which are still working for the actual crack width. For the distribution function $\Phi(w)$ any monotonicly decreasing function with $\Phi(w = 0) = 1$ and $\Phi(w \rightarrow \infty) = 0$ can be chosen. This leads to:

$$\sigma_{ct}(w) = f_{ct} \cdot \Phi(w)$$

## MODEL FOR A MONOTONICLY INCREASING CRACK WIDTH

In the first stage of a *σ-w relation* a steep decrease of the transferable stress occurs. If the stress went down to about one third of $f_{ct}$, the following decrease is considerably more flat. The first branch may be interpreted as successive failure of the primary bearing mechanism, that is the adhesion between grain and matrix the second branch as secondary bearing mechanism, that is friction between grain and matrix after cracking.

These two load transferring mechanisms can be described by two groups of friction blocks which are arranged in parallel as shown in figure 1. Friction block $A$ symbolizes the $n_a$ friction blocks of the first group, friction block $B$ the $n_b$ friction blocks of the second group. Group $A$ describes the primary bearing mechanism, group $B$ the secondary one.

The transferred stresses for $w = 0$ will be designated with $\sigma_{a,0}$ or $\sigma_{b,0}$ respectively.

$$\sigma_{ct}(w) = \sigma_{a,0} \cdot \Phi_a(w) + \sigma_{b,0} \cdot \Phi_b(w)$$

The distribution function $\Phi_a(w)$ must describe the sudden failure of the primary bearing mechanism. The following formulation was chosen: $\Phi_a(w) = e^{-(w/w_a)^2}$ The slower failure of the elements of group $B$ is described by: $\Phi_b(w) = e^{-w/w_b}$. This leads to:

$$\sigma_{ct}(w) = \sigma_{a,0} \cdot e^{-(w/w_a)^2} + \sigma_{b,0} \cdot e^{-w/w_b} \tag{1}$$

The adoption of the function given in equation (1) to experimental results, though, is easily possible. In figure 3, the procedure is illustrated using experimental results from (1). The sum resulting from $\sigma_{a,0}$ and $\sigma_{b,0}$ is equal to the tensile strength of concrete $f_{ct}$, at which $\sigma_{a,0}$ represents the first steep decrease of the *$\sigma$-$w$ relation* and $\sigma_{b,0}$ the second flat branch. Both crack widths $w_a$ and $w_b$ are equal to those where $1/e \approx 37\%$ of the formerly existing friction blocks are still working.

## MODEL FOR CYCLIC LOADING

The material model introduced above will be extended for cyclic loading. A material model is applicable for cyclic loading only if it can describe all kind off different deformation histories (lower stress $= 0$, $> 0$, $< 0$ or variable) in a sensible way.

### Description of Hysteresis Loops with Rheological Elements

Hysteretic material behavior can be described by the combination of 2 friction blocks and 2 springs shown in figure 4 (left). If this mechanism is forced to a certain displacement, the relation between load and deformation shown in figure 4 (right) will be obtained.

The load $P$ remains constant until point $\boxed{2}$, when the direction of deformation reverses. Between point $\boxed{1}$ and point $\boxed{2}$ this leads to:

$$P = P_1 + P_2; \quad w_1 = w - \frac{P}{K_1}; \quad w_2 = w_1 - \frac{P_2}{K_2} \tag{2}$$

A crack does not open until the tensile strength of concrete is reached, i.e. for the material model that even for a crack width of $w = 0$ the mechanism must transfer the maximum load. This is achieved by adjusting the displacements $w_1$ and $w_2$ at point $\boxed{1}$ in such a way that both friction blocks $R_1$ and $R_2$ just transfer the maximum friction force $P_1$ or $P_2$ respectively. At the point $\boxed{1}$ this leads to:

$$P = P_1 + P_2; \quad w = 0; \quad w_1 = -\frac{P}{K_1}; \quad w_2 = w_1 - \frac{P_2}{K_2} \tag{3}$$

After reversal of the deformation direction at point $\boxed{2}$ spring $F_1$ is compressed until the friction force at $R_1$ is equal to $-P_1$. By that point $\boxed{3}$ is reached, since which friction block $R_1$ slides in negative direction. Between point $\boxed{2}$ and point $\boxed{3}$ both friction blocks are motionless.

Between point $\boxed{2}$ and point $\boxed{3}$ this leads to:

$$P = (w - w_1) \cdot K_1; \quad w_1 = w_1\boxed{2} \;; \quad w_2 = w_2\boxed{2} \tag{4}$$

where: $w_{1,2}\boxed{2}$ = displacement $w_{1,2}$ at point $\boxed{2}$

The stiffness of spring $F_2$ is adjusted in a way that on achievement of a crack width $w = 0$ the point $\boxed{4^*}$ on the load axis will be reached. Point point $\boxed{4^*}$ has the ordinate $-2 \cdot P_1$. This leads to:

$$\left.\begin{array}{rcl} P(w=0) & = & -2 \cdot P_1 \\ w & = & 0 \end{array}\right\} \Rightarrow K_2 = \frac{P_1 + P_2}{w_{max} - \frac{3 \cdot P_1 + P_2}{K_1}} \tag{5}$$

where $w_{max}$ = maximum value of $w$ ever reached in the deformation history Between point $\boxed{3}$ and point $\boxed{4}$ the displacement $w_1$ results from equilibrium for friction block 1, block 2 is motionless. This leads to:

$$P = (w - w_1) \cdot K_1; \quad w_1 = \frac{w_2 \cdot K_2 + w \cdot K_1 + P_1}{K_1 + K_2}; \quad w_2 = w_2\boxed{2} \tag{6}$$

At point $\boxed{4}$ the deformation direction is changed again. Spring $F_1$ is now elongated until friction block $R_1$ starts sliding in positive direction at point $\boxed{5}$. Both friction blocks $R_1$ and $R_2$ are up to point $\boxed{5}$ motionless. In case the crack width was zero when the deformation direction was reversed, point $\boxed{4*}$ has been reached on the load axis. Point $\boxed{5*}$ will then be precisely on the $w$-axis. Between point $\boxed{4}$ and point $\boxed{5}$ this leads to:

$$P = (w - w_1) \cdot K_1; \quad w_1 = w_1\boxed{4} \;; \quad w_2 = w_2\boxed{2} \tag{7}$$

From point $\boxed{5}$ on, $R_1$ slides in positive direction until point $\boxed{6}$ is reached, which is the starting point $\boxed{2}$ of the hysteresis loop. The displacement $w_1$ results again out of equilibrium for $R_1$. Between point $\boxed{5}$ and point $\boxed{6}$ this leads to:

$$P = (w - w_1) \cdot K_1; \quad w_1 = \frac{w_2 \cdot K_2 + w \cdot K_1 - P_1}{K_1 + K_2}; \quad w_2 = w_2\boxed{2} \tag{8}$$

From point $\boxed{6}$ on both friction blocks slide in positive direction. Now the same relations as between point $\boxed{1}$ and point $\boxed{2}$ are valid again.

If the deformation direction will later be changed again, more hysteresis loops will result. By definition of the spring stiffness $K_2$ in equation (5) in every hysteresis loop the point $\boxed{4*}$ with the coordinates ($w = 0$ , $P = -2 \cdot P_1$) will be reached, if the deformation $w = 0$ is achieved.

## Extension of the Material Model for Cyclic Loading

By the above-mentioned arrangement of elements, shown in figure 4, a hysteresis behavior of material can be described very well. The model presented in figure 1 will be extended as shown in figure 2. Each of the friction blocks of the element groups $A$ and $B$ will be replaced by a combination of 2 friction blocks and 2 springs. Additionally, element group $C$ will be introduced.

Element Groups A and B. Equations 3 to 8 describe the reaction of a mechanism consisting of the series arrangement of two springs and two friction blocks. The mechanisms $A$ and $B$ shown in figure 2 symbolize the $n_a$ or $n_b$, respectively, parallel arranged similar mechanisms, of which the element group consists of. According to the explanations in chapter 'Fundamentals' the behavior of a element group can be described combining equations 3 to 8 with the distribution functions. The behavior of the element groups $A$ and $B$ is shown in figure 5 and in figure 6. The statements for the distribution functions $\Phi_a(w)$ and $\Phi_b(w)$ will be taken over from the model for monotonic loading. The basic idea of the distribution functions is that they state how many of the originally existing elements are working at the actual crack width. During opening of the crack the elements fail and will not be reactivated when the crack is closed again. The distribution functions must therefore refer to the maximum crack width ($=w_{max}$) ever achieved in the deformation history.

$$\Phi_a(w) = e^{-(w_{max}/w_a)^2} \qquad \Phi_b(w) = e^{-w_{max}/w_b} \tag{9}$$

Element Group $C$. A crack, once having opened, cannot be closed again. During the opening of the crack, small particles separate from the crack surface. These particles dislocate while the crack is open and prevent the surfaces from fitting into each other again without constraint. This behavior of the material is modeled by element group $C$. Element group $C$ is only active within the hysteresis loop. It was chosen:

$$\sigma_c(w) = \begin{cases} -\sigma_{c,0} \cdot (\frac{w_{max}}{w} - 1) & \text{for negative deformation direction} \\ -\sigma_{c,0} \cdot (\frac{w_{max}}{w} - 1) \cdot C_2 & \text{for positive deformation direction} \end{cases} \tag{10}$$

The behavior of element group $C$ as described by equations 10 is plotted in figure 7. The constant $C_2$ influence the area within the hysteresis loop of element group $C$.

Total Behavior of the Material Model. The behavior of the total material model results from the addition of the stresses of the three element groups $A$, $B$ and $C$. In figure 8 this addition was carried out for the hysteresis loops plotted in figures 5 - 7.

## COMPARISON OF EXPERIMENTAL RESULTS AND THE MATERIAL MODEL

The best way to compare the material model and experiments are centric tension tests. It must be taken into consideration that every experimental curve starts with an initial slope. A measurement is possible over a gauge length only. The values measured include the crack width and the bulk deformation. The bulk deformation can be determined as sufficiently exact assuming linear elastic behavior. Figure 9 shows an experimental result from (4) and figure 10 the result from the material model. In order to adopt the model calculation to the experimental results the parameters $\sigma_{a,0}$, $\sigma_{b,0}$, $w_a$ and $w_b$, which describe the envelope of the curve have been read from the experimental curve as described in figure 3.

## SUMMARY

A material model describing the behavior of tension loaded concrete has been introduced. The model consist of simple rheological elements. Compared with other formulations known form literature the following advantages result:

- The material model is based on a mechanic clear and reproducible conception.
- The envelope and the hysteresis loops are described by one and the same model. It is not necessary as e.g. in (5) to work with a totally different definition at the envelope and within the hysteresis loops.
- The model only works with information which is available at the actual state of loading. It is not necessary for the description of the hysteresis loop that, at the beginning of a hysteresis loop, the later achieved lower stress is known, as in (4).
- The model can easily be adopted to other concrete strengths or mixtures. It is applicable to normal, light and high strength concrete.

- The relation between stress and crack width is described by very simple analytic relations. Because of this the model can be easily used together with Finite Element Codes. The model was implemented into the FE-program SNAP (3). Recalculating some own experiments and experiment from the literature very good results have been achieved.
- The $\sigma$-$w$ *relation* is integrable. The fracture energy $G_f$ is herewith easy to calculate.

The model introduced allows a realistic description of the complex behavior of concrete loaded by tension. This model can always be applied if concrete is loaded perpendicular to the crack. An extension to load perpendicular to and parallel with the crack is worded at.

## SYMBOLS USED

$\sigma$ = stress (MN/m$^2$)

$\sigma_{ct}$ = tensile stress (MN/m$^2$)

$f_{ct}$ = tensile strength (MN/m$^2$)

$w$ = crack width (m)

$K$ = spring rate (N/m)

$P$ = force (N)

$R$ = friction force (N)

## REFERENCES


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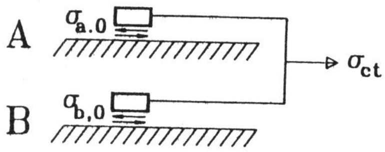


Figure 1 Material model 1

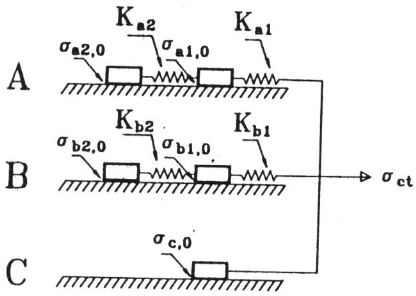


Figure 2 Material model for cyclic loading

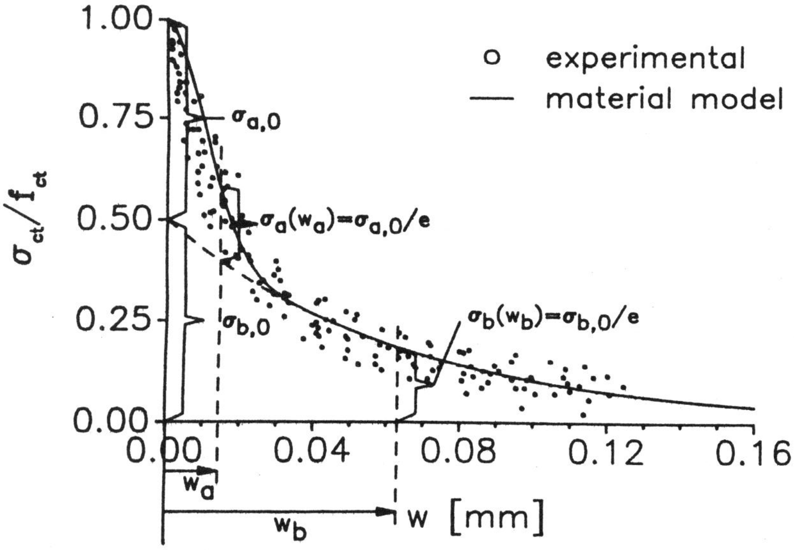


Figure 3 $\sigma$-$w$ relation fitted to experimental results

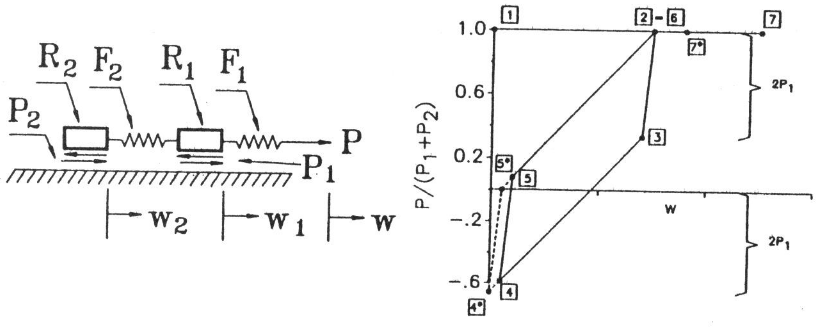


Figure 4 Hysteresis-model and hysteresis model behavior

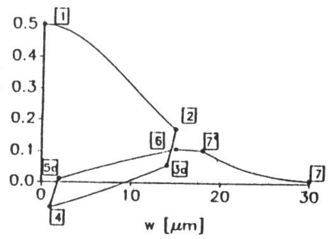


Figure 5 Hysteresis loop of element group $A$

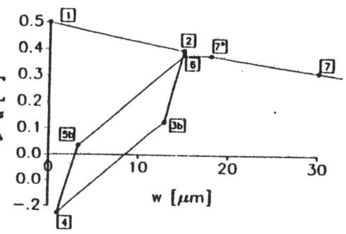


Figure 6 Hysteresis loop of element group $B$

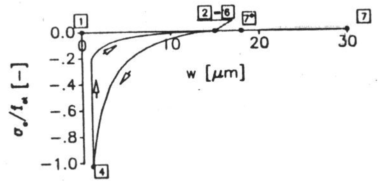


Figure 7 Hysteresis loop of group $C$ element

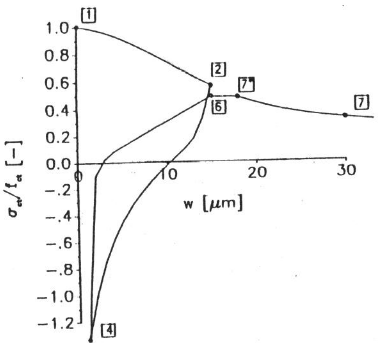


Figure 8 Hysteresis loop of material model

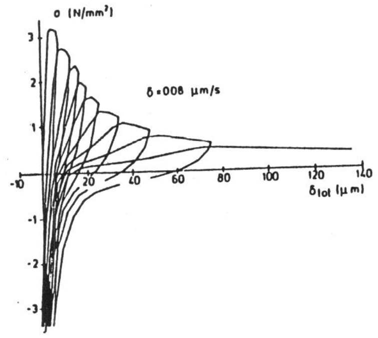


Figure 9 Experiment - minimum stress $\approx -f_{ct}$

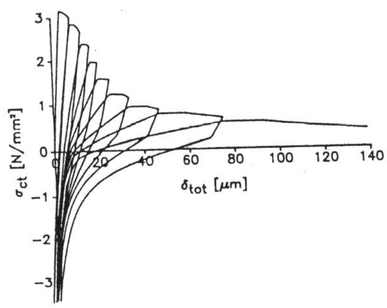


Figure 10 Material model - minimum stress $\approx -f_{ct}$