# STRAIN HARDENING EFFECTS IN DUCTILE POROUS METALS

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The most classical model for the mechanical behaviour of ductile porous metals was formulated by GURSON (1,2). While retaining the general functional form of the yield criterion involved in that model, this paper suggests some improvements of the formulation of strain hardening effects initially proposed by Gurson.

## 1. INTRODUCTION

The well-known GURSON (1,2) yield criterion for ductile porous metals reads :

$$\Phi(\Sigma^{eq},\Sigma^{m},f,\bar{\sigma}) = \left(\frac{\Sigma^{eq}}{\bar{\sigma}}\right)^2 + 2f\cosh\left(\frac{3}{2}\frac{\Sigma^{m}}{\bar{\sigma}}\right) - 1 - f^2 = 0 \qquad (1)$$

In this equation, strain hardening is assumed to be isotropic and described by the parameter $\bar{\sigma}$. $\bar{\sigma}$ is given by the hardening law of the sound matrix $\bar{\sigma} = \sigma^Y(\bar{\varepsilon})$ , where $\bar{\varepsilon}$ is a cumulated strain obeying the following evolution equation :

$$(1-f)\ \dot{\bar{\varepsilon}}\ \sigma^Y(\bar{\varepsilon}) = \Sigma_{ij} D^p_{ij} \qquad (2)$$

This approach was extended to the case of kinematic hardening by MEAR and HUTCHINSON (3). The evolution equation for the centre of the yield locus proposed by these authors is such that the predictions of the model be identical to those of the isotropic Gurson model under proportional stressing.

In this paper, two drawbacks of Gurson's formulation of hardening effects will first be exhibited. New versions of the Gurson model that

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overcome these drawbacks will then be proposed for both isotropic and kinematic hardenings. Finally, some comparisons between the predictions of the various models (Gurson, Mear and Hutchinson and those proposed here) and numerical simulations of porous media will be given in order to evidence the improvements brought.

## 2. THE HOLLOW PLASTIC SPHERE UNDER HYDROSTATIC LOADING

We begin by reminding the classical solution of the problem of a hollow rigid-plastic sphere subjected to hydrostatic tension ; this solution will play an important role in the sequel. The inner and outer radii are denoted a and b respectively. The material is assumed to obey the Mises criterion and the associated flow rule. Hardening is of isotropic type and described by $\sigma^Y = \sigma^Y(\varepsilon^{eq})$ , where $\sigma^Y$ is the current yield stress and $\varepsilon^{eq}$ the equivalent cumulated Mises strain.

It follows from the incompressibility of the matrix that $r^3(t)-r^3(0) = a^3(t)-a^3(0) \equiv \omega(t)$, where r(t) is the Eulerian radial coordinate of any material point. Time-integration of the expression of the equivalent strain rate yields then (since $\dot{\omega} > 0$ for a tensile loading) :

$$\varepsilon^{eq}(r(t),t) = \int_0^t \frac{2}{3}\frac{\dot{\omega}(\tau)}{r^3(\tau)}\,d\tau = \frac{2}{3}\ln\left(\frac{r^3(t)}{r^3(t)-\omega(t)}\right) \tag{3}$$

The macroscopic yield stress under hydrostatic tension $\Sigma_Y^m = \sigma_{rr}(b(t))$ is then easily obtained by integrating the equilibrium equation $d\sigma_{rr}/dr = 2(\sigma_{\theta\theta} - \sigma_{rr})/r = 2\sigma^Y(\varepsilon^{eq}(r,t))/r$ together with the boundary condition $\sigma_{rr}(a(t)) = 0$ :

$$\Sigma_Y^m = \int_{a^3(t)}^{b^3(t)} \frac{2}{3}\,\sigma^Y\left(\frac{2}{3}\ln\frac{r^3}{r^3-\omega(t)}\right)\frac{dr^3}{r^3} \tag{4}$$

The value of $\Sigma_Y^m$ is the same if hardening is of kinematic type, since the stressing history is proportional.

## 3. TWO DRAWBACKS OF GURSON'S FORMULATION OF STRAIN HARDENING

The macroscopic Gurson model was initially obtained through approximate homogenization of the microscopic behaviour of a typical "porous" geometry, namely a hollow rigid-plastic sphere subjected to an *axisymmetric* loading. As such, this model should reproduce the well-known exact solution in the case of a *spherically* symmetric loading. However, we are going to show that for such a spherically strained sphere :

i. the macroscopic yield stress under hydrostatic loading $\Sigma^m_{Y,GURSON}$ obtained from the Gurson model is different from the exact one (eq. (4)) ;

ii. the yield stresses under purely deviatoric ($\Sigma_Y^{eq}$) and purely hydrostatic ($\Sigma_Y^m$) loadings cannot be expressed in terms of a single parameter $\bar{\sigma}$ as predicted by eq.(1).

Proof of i. For a hollow sphere under hydrostatic loading, $D = D^m\cdot 1$ and $\Sigma = \Sigma_Y^m\cdot 1$ , with $D^m = \dot{f}/3(1-f)$ and $\Sigma_Y^m$ given by (1) with $\Sigma^{eq}=0$ , i.e. $\Sigma_Y^m = -\frac{2}{3}\bar{\sigma}\ln f$ ; hence eq.(2) yields

$$\dot{\bar{\varepsilon}} = -\frac{2}{3}\frac{\ln f}{(1-f)^2}\dot{f} \quad \Rightarrow \quad \bar{\varepsilon} = \frac{2}{3}\left(-\frac{f\ln f}{1-f} + \frac{f_0\ln f_0}{1-f_0} + \ln\frac{1-f_0}{1-f}\right)$$

In fact this can be shown, using eq.(3), to be the exact average value of $\varepsilon^{eq}$ over the matrix. Hence we get for such a hollow sphere :

$$\Sigma^{m}_{Y,GURSON} = -\frac{2}{3}\sigma^{Y}(\bar{\varepsilon})\ln f \equiv \left(\frac{2}{3}\int_{a^3(t)}^{b^3(t)}\frac{dr^3}{r^3}\right)\cdot\sigma^{Y}\left(\int_{a^3(t)}^{b^3(t)}\frac{2}{3}\ln\frac{r^3}{r^3-\omega(t)}\frac{dr^3}{b^3(t)-a^3(t)}\right),$$

which is clearly different from eq.(4). A detailed study of the gap between these formulae shows that it increases with decreasing porosity and increasing hardening slope.

The Mear-Hutchinson model for the kinematic hardening case yields the same value of $\Sigma^{m}_{Y}$ as Gurson's model and is therefore subject to the same drawback.

<u>Proof of ii.</u> First, note that $\sigma^{Y}(\varepsilon^{eq})$ being an increasing function of r,

$$\Sigma^{m}_{Y} > -\frac{2}{3}\ln f\,\langle\sigma^{Y}\rangle_{V^m} \tag{5}$$

where $\langle\sigma^{Y}\rangle_{V^m}$ is the average value of $\sigma^{Y}$ over the matter volume $V^m$. Second, suppose that the sphere, after having been strained hydrostatically, is subjected to a purely deviatoric loading. Then

$$\Sigma^{eq}_{Y} = \|\Sigma'\| = \|\langle\sigma'\rangle_V\| \leqslant \langle\|\sigma'\|\rangle_V \leqslant (1-f)\,\langle\sigma^{Y}\rangle_{V^m} \tag{6}$$

where $\sigma'$ and $\Sigma'$ denote the microscopic and macroscopic stress deviators, respectively, $\|\ \|$ the Von Mises norm, and V the total volume (matter plus void). Now, use of GURSON's criterion (1) yields $\Sigma^{m}_{Y,GURSON} = -\frac{2}{3}\bar{\sigma}\ln f$ and $\Sigma^{eq}_{Y,GURSON} = (1-f)\bar{\sigma}$ ; if these expressions were equal to $\Sigma^{m}_{Y}$ and $\Sigma^{eq}_{Y}$, then $\bar{\sigma}$ would be greater than $\langle\sigma^{Y}\rangle_{V^m}$ by eq.(5), and smaller by eq.(6), contradiction.

## 4. NEW FORMULATIONS OF STRAIN HARDENING EFFECTS

Let us consider the isotropic case first. In order to overcome difficulty <u>ii</u>, we propose to introduce two distinct parameters $\Sigma_1$ , $\Sigma_2$ instead of Gurson's single $\bar{\sigma}$ ; the criterion reads now

$$\Phi(\Sigma^{eq},\Sigma^{m},f,\Sigma_1,\Sigma_2) = \left(\frac{\Sigma^{eq}}{\Sigma_1}\right)^2 + 2f\,\mathrm{ch}\left(\frac{3}{2}\frac{\Sigma^{m}}{\Sigma_2}\right) - 1 - f^2 = 0$$

$\Sigma_1$ and $\Sigma_2$ are supposed to depend on two strain parameters $E^{eq}$, $E^{m}$ defined by $\dot{E}^{eq} = \left(\frac{2}{3}D'^{p}_{ij}D'^{p}_{ij}\right)^{1/2}$ ($D'^{p} \equiv$ deviator of $D^{p}$) and $\dot{E}^{m} = |D^{p}_{m}|$, which measure the amount of deviatoric and hydrostatic hardenings, respectively.

Defining $\Sigma_1(E^{eq},E^{m})$ and $\Sigma_2(E^{eq},E^{m})$ is equivalent to defining $\Sigma^{eq}_{Y}(E^{eq},E^{m})$ and $\Sigma^{m}_{Y}(E^{eq},E^{m})$. This is done through an approximate evaluation of the yield stresses under purely deviatoric and purely hydrostatic loadings of a hollow sphere strained axisymmetrically and proportionally : $\dot{E}^{m}/\dot{E}^{eq} = C^{st}$. The following approximations are made :

*(a)* The velocity field is assumed to be the sum of a spherically symmetric $1/r^2$ radial field plus a uniform deviatoric field, in accordance with Rice and Tracey's findings (4).

*(b)* The geometry changes due to the deviatoric field are neglected.

*(c)* $\Sigma_Y^{eq}$ and $\Sigma_Y^m$ are supposed to be given by the same formulae (6) (with "=" instead of "≤") and (4) as in the purely spherical case, except for the replacement of $\sigma^Y(\varepsilon^{eq})$ by its average value over the infinitesimally thin shell of radius r, $\langle \sigma^Y(\varepsilon^{eq}) \rangle_r$ :

$$\Sigma_Y^{eq} = \frac{1}{b^3(t)} \int_{a^3(t)}^{b^3(t)} \langle \sigma^Y(\varepsilon^{eq}) \rangle_r dr^3 \quad , \quad \Sigma_Y^m = \int_{a^3(t)}^{b^3(t)} \frac{2}{3} \langle \sigma^Y(\varepsilon^{eq}) \rangle_r \frac{dr^3}{r^3}$$

*(d)* $\langle \sigma^Y(\varepsilon^{eq}) \rangle_r \simeq \sigma^Y\left(\langle \varepsilon^{eq} \rangle_r\right)$ and $\langle \varepsilon^{eq} \rangle_r \simeq \int_0^t \left\langle \dot{\varepsilon}^{eq^2} \right\rangle_r^{1/2} dt.$

The problem is then reduced to calculating $\left\langle \dot{\varepsilon}^{eq^2} \right\rangle_r^{1/2}$, which is found to be

$\left\langle \dot{\varepsilon}^{eq^2} \right\rangle_r^{1/2} = \left( \frac{4}{r^6} b^6(t) \dot{E}^{m^2} + \dot{E}^{eq^2} \right)^{1/2}$ ; the analytical time integration follows.

The value of $\Sigma_Y^m$ obtained in this way coincides with the exact formula (4) in the purely spherical case ($\dot{E}^{eq} = 0$) since in this case, all approximations are exact. This solves difficulty <u>**ii**</u>.

In the kinematic case, the criterion is supposed to be of the (quite sensible) form proposed by Mear and Hutchinson :

$$\Phi(\Sigma', \Sigma^m, f, A', A^m) = \left(\frac{\Sigma^{eq}}{\sigma_0}\right)^2 + 2f \cosh\left(\frac{3}{2}\frac{\Sigma^m - A^m}{\sigma_0}\right) - 1 - f^2 = 0,$$

$$\Sigma^{eq} = \left[\frac{3}{2}(\Sigma'_{ij} - A'_{ij})(\Sigma'_{ij} - A'_{ij})\right]^{1/2}.$$

The evolution laws for A' and $A^m$ are supposed to be of the form :

$$\check{A}'_{ij} = H(E^{eq}, E^m)\, D'^p_{ij} \quad ; \quad \dot{A}^m = K(E^{eq}, E^m)\, D^p_m$$

where $\check{}$ denotes the Jaumann derivative. The values of H and K are again obtained through an approximate analysis of a hollow sphere strained axisymmetrically and proportionally. The formulae obtained are somewhat too complex to be shown here. They are again exact in the case of a purely spherical straining.

## <u>5. NUMERICAL SIMULATIONS OF THE STRAINING OF A HOLLOW SPHERE</u>

The axisymmetric, proportional straining of an elasto-plastic hollow sphere has been simulated numerically. The initial porosity of the sphere is 0.1176 %. Four kinds of hardening types are considered : isotropic or kinematic, and non-linear or linear hardening law. The results of the computations are compared with the predictions of the various models. Figures 1 through 4 show the mean stress, as a function of the mean strain.

In the case of isotropic hardening, the comparison shows that the model proposed here is a little better than that of Gurson for non-linear hardening *(Fig. 1)*, and much better for linear hardening *(Fig. 2)*.

The same conclusions apply to the kinematic case *(Figs. 3 and 4)*.

## 6. NUMERICAL SIMULATIONS OF PERIODIC POROUS MEDIA

Until now, this paper was concerned with materials containing a single cavity. In order to account for the effects of interactions between the numerous voids of a porous metal, TVERGAARD (5) proposed to multiply the porosity f by an empirical factor q, ranging from 1.25 to 1.5.

KOPLICK and NEEDLEMAN (6) performed numerical simulations of periodic porous materials with hardening law given by $\sigma^Y = \sigma_0 \left(1 + E\varepsilon^{eq}/\sigma_0\right)^n$. Looking for the value of q that offers the best fit with Gurson's model, they found that q was a function of the hardening exponent n. It may be suspected that this dependence is a mere consequence of Gurson's imperfect description of strain hardening, rather than a real effect.

Indeed, Figure 5 shows that, in the framework of the model proposed here, the best value of q is a constant, $q \simeq 1.35$. This value also agrees quite well with that ($q = 4/e \simeq 1.47$) derived by PERRIN and LEBLOND (7) by a *self-consistent* method.

### REFERENCES


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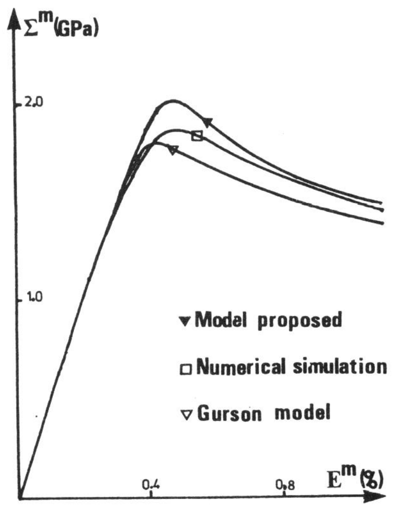


Figure 1
Isotropic, non-linear hardening
$\dot{E}^m/\dot{E}^{eq} = 0.3$

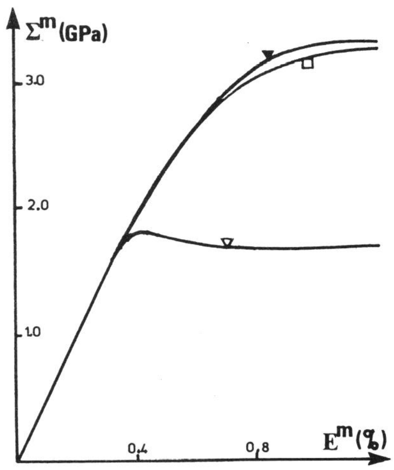


Figure 2
Isotropic, linear hardening
$\dot{E}^m/\dot{E}^{eq} = 0.3$

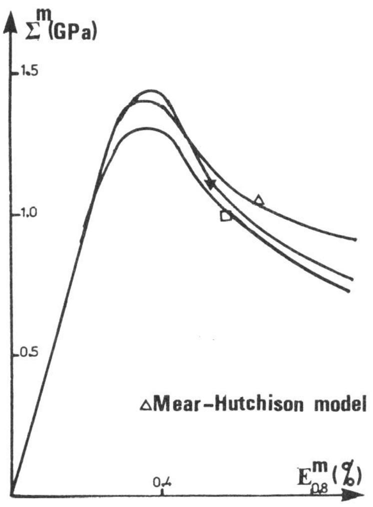


Figure 3
Kinematic, non-linear hardening
$\dot{E}^m/\dot{E}^{eq}$ = 0.05

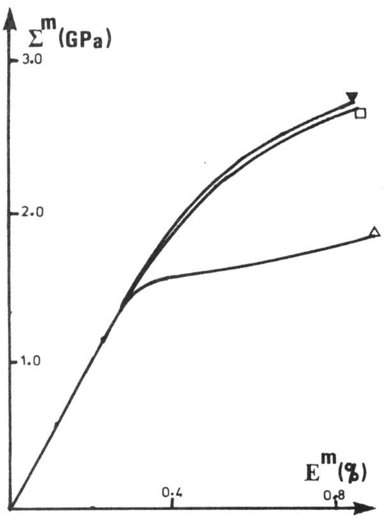


Figure 4
Kinematic, linear hardening
$\dot{E}^m/\dot{E}^{eq}$ = 0.05

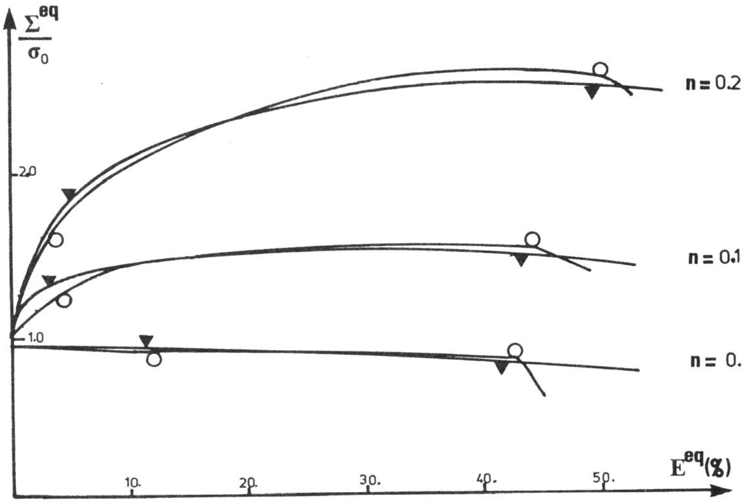


Figure 5
Comparisons between the Koplick and Needleman numerical simulations of periodic porous media (○) and the predictions of the model proposed (▼), for q = 1.35.