THE CRACK-INCLUSION INTERACTION PROBLEM IN AN INFINITE STRIP

O.S. Yahşi* and Ş. Abbas*

The problem of interaction between a crack and an inclusion in an infinite strip is considered. The Green's functions for a pair of dislocations and a pair of concentrated body forces are used to generate the crack and the inclusion in the infinite strip which is assumed to be elastic continuum, and to which uniaxial tension is applied away from the crack-inclusion interaction region. The problem is reduced to a system of three integral equations having Cauchy-type dominant kernels. The stress intensity factors are calculated and tabulated for various crack-inclusion and strip geometries and the inclusion to matrix modulus ratios.

INTRODUCTION

In studying the fracture of multi-phase materials it is often necessary to take into account among other factors, the effect of imperfections in the material. From the viewpoint of fracture mechanics two important classes of imperfections are the planar flaws which may be idealized as cracks and relatively thin inhomogeneities which may be idealized as flat inclusions with "sharp" boundaries.

All of the crack-inclusion interaction problems solved by Erdoğan and Gupta (1), Erdoğan et al (2), Erdoğan and Xue-Hui (3), Atkinson (4) assume continuity of displacements at the interface of the inclusion and in all of these studies in-plane dimension of the medium are assumed to be large compared to the lengths of and the distance between the crack and the inclusion so that the effect of the remote boundaries on the perturbed stress state may be neglected.

In this paper a crack-inclusion interaction problem in an infinite strip is considered. This "flat" inclusion is represented by a membrane with no bending stiffness. The basic Green's functions for a pair of dislocation and a pair of concentrated body forces are used to generate the crack and the inclusion in the

* Department of Mechanical Engineering Middle East Technical University, Ankara, Turkey.

infinite strip on which a uniform tensile stress, σ_o, is acted far from the crack-inclusion interaction region.

INTEGRAL EQUATIONS OF THE PROBLEM

The geometry of crack-inclusion interaction problem is shown in Figure 1. It is assumed that the strip is under a state of plane strain or generalized plane stress. It is further assumed that the inclusion is sufficiently thin so that its bending stiffness may also be neglected.

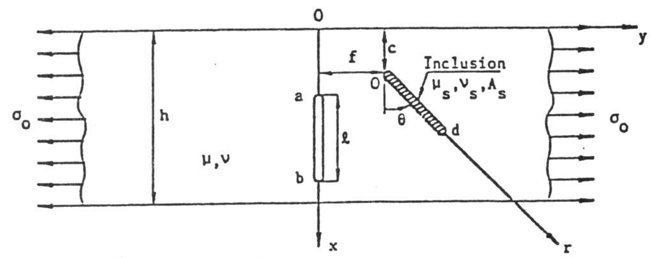

Figure 1 Geometry of infinite strip with crack-inclusion interaction.

Referring to Fig.1, consider the stresses and displacements due to a pair of point dislocations on the x axis, a pair of concentrated forces on the line θ = constant, and general solution for an infinite strip under the effect of uniform tensile stress σ_o. Let σ_{dij}, σ_{pij}, σ_{sij}, (i, j) = (x, y) or (i, j) = (r, θ) be the stress components due to dislocations, concentrated forces and strip, respectively. The total stress state in the elastic strip may therefore, be expressed as

$$\sigma_{ij}(x, y) = \sigma_{dij}(x, y) + \sigma_{pij}(x, y) + \sigma_{sij}(x, y) \qquad (i, j) = (x, y) \tag{1}$$

By assuming that the dislocations are distributed along $a < x < b$, $y = 0$ forming a crack, for a pair of point dislocations with densities g and h defined by

$$\frac{\partial}{\partial x}\left[v\,(x, +0) - v\,(x, -0)\right] = g(x), \qquad a < x < b\ , \tag{2 a}$$

$$\frac{\partial}{\partial x}\left[u\,(x, +0) - u\,(x, -0)\right] = h(x), \qquad a < x < b\ , \tag{2 b}$$

the stress sate in an infinite plane may be expressed as

$$\sigma_{dij}(x, y) = \frac{2\mu}{\pi\,(\kappa + 1)} \int_a^b \left[G_{ij}(t, y, x)\, g(t) + H_{ij}(t, y, x)\, h(t)\right] dt\ ,$$
$$(i, j = x, y) \tag{3}$$

where the expressions of G_{ij} and H_{ij} are given by Abbas (5) and μ is the shear modulus and $\kappa = 3 - 4\upsilon$ for plane strain and $\kappa = (3 - \upsilon) / (1 + \upsilon)$ for generalized plane stress, υ being the Poisson's ratio.

Similarly, by using the stress expressions due to a pair of concentrated forces P_x and P_y given in (5) and by assuming that the inclusion is located along the line $0 < r < d$, θ = constant and finally by using the following continuity and equilibrium conditions for the inclusion

$$u_s(r, \theta + 0) = u_s(r, \theta - 0), \quad (s = r, \theta), \quad 0 < r < d, \tag{4 a,b}$$

$$\sigma_{\theta\theta}(r, \theta + 0) - \sigma_{\theta\theta}(r, \theta - 0) = 0, \quad 0 < r < d, \tag{4 c}$$

$$-p(r) = \sigma_{r\theta}(r, \theta + 0) - \sigma_{r\theta}(r, \theta - 0), \quad 0 < r < d, \tag{4 d}$$

the stress components σ_{pij} can be found as a function of p(r). Their expressions are given in (5).

Let us now consider the stress state in an infinite strip, $0 < x < h$ parallel to the y axis and loaded at infinity by uniform tensile stress σ_o. Using Fourier transforms it may be shown that

$$\sigma_{sxx}(x,y) = \frac{\mu}{\pi}\int_{-\infty}^{\infty}\left\{ -\left[|\alpha|\left(E_1(\alpha) + xE_2(\alpha)\right) - \frac{\kappa - 1}{2}E_2(\alpha)\right]e^{-|\alpha|x} \right.$$

$$\left. + \left[|\alpha|\left(E_3(\alpha) + xE_4(\alpha)\right) + \frac{\kappa - 1}{2}E_4(\alpha)\right]e^{|\alpha|x}\right\} e^{-i\alpha y}\, d\alpha$$

$$0 \leq x \leq h \qquad -\infty < y < \infty \tag{5 a}$$

$$\sigma_{sxy}(x,y) = \frac{i\mu}{\pi}\int_{-\infty}^{\infty}\left\{ -\left[|\alpha|\left(E_1(\alpha) + xE_2(\alpha)\right) - |\alpha|/\alpha\,\frac{\kappa + 1}{2}E_2(\alpha)\right]e^{-|\alpha|x} \right.$$

$$\left. + \left[-|\alpha|\left(E_3(\alpha) + xE_4(\alpha)\right) - |\alpha|/\alpha\,\frac{\kappa + 1}{2}E_4(\alpha)\right]e^{|\alpha|x}\right\} e^{-i\alpha y}\, d\alpha$$

$$0 \leq x \leq h \qquad -\infty < y < \infty \tag{5 b}$$

where E_1, E_2, E_3 and E_4 are unknown function of α and are determined from the stress free boundary conditions at x= 0 and x = h. Thus, if one substitutes σ_{pij} expressions, (3), and (5) into (2) and the resulting equation into the boundary conditions

$$\sigma_{xx}(0, y) = \sigma_{xy}(0, y) = \sigma_{xx}(h, y) = \sigma_{xy}(h, y) = 0 \tag{7 a-c}$$

the unknown functions E_1, E_2, E_3 and E_4 can be expressed in terms of the unknown functions p, h and f.

Thus, the complete solution of the problem will be obtained once the unknown functions g, h and p are determined. These unknown functions may be determined by expressing the stress boundary conditions on the crack surfaces and the displacement compatibility condition along the inclusion, namely

$$\sigma_{yy}(x, 0) = 0\,, \qquad \sigma_{xy}(x, 0) = 0\,, \qquad a < x < b \qquad (8\text{ a, b})$$

$$\varepsilon_{rr}(r, \theta) = \varepsilon_i\,(r)\,, \qquad 0 < r < d \qquad (8\text{ c})$$

where $\varepsilon_i\,(r)$ is the longitudinal strain in the inclusion and it can be expressed as (5)

$$\varepsilon_i\,(r) = -\frac{1 + \kappa_s}{8\,\mu_s\,A_s}\int_r^d p(r_o)\,dr_o \qquad (8)$$

where μ_s and κ_s are the elastic constants and A_s is the cross-sectional area of the inclusion.

Finally, by substituting (2) into (7 a-c) the three integral equations of the problem may be obtained as follows:

$$\int_a^b \frac{g(t)}{t - x}\,dt + \int_a^b k_{11}\,(t, x)\,g(t)\,dt + \int_o^d k_{13}\,(r_o, x)\,p(r_o)\,dr_o$$

$$= -\frac{\pi\,(\kappa + 1)}{2\,\mu}\,\sigma_o\,, \qquad a < x < b\,, \qquad (9)$$

$$\int_a^b \frac{h(t)}{t - x}\,dt + \int_a^b k_{22}\,(t, x)\,h(t)\,dt + \int_o^d k_{23}\,(r_o, x)\,p(r_o)\,dr_o = 0\,,$$
$$a < x < b\,, \qquad (10)$$

$$\int_o^d \frac{p(r_o)}{r_o - r}\,dr_o + \int_a^b k_{31}\,(t, r)\,g(t)\,dt + \int_a^b k_{32}\,(t, r)\,h(t)\,dt$$

$$+\int_a^b k_{33}\,(r_o, r)\,p(r_o)\,dr_o = -\frac{\pi\,(\kappa + 1)^2}{4\,\kappa}\,\sigma_o\,(\sin^2\theta + \frac{\kappa - 3}{\kappa + 1}\cos^2\theta)$$
$$0 < r < d\,, \qquad (11)$$

where the expressions of k_{ij} (i, j = 1, 2, 3) are given in (5). These integral equations are subject to the following continuity and static equilibrium conditions

$$\int_a^b g(t)\, dt = 0\,, \quad \int_a^b h(t)\, dt = 0\,, \quad \int_o^d p(r_o)\, dr_o = 0 \tag{12 a-c}$$

Thus the system of singular integral equations must be solved under the conditions (12 a-c). By using function-theoretic method it can be shown that the unknown functions g, h and p are of the following form (5):

$$g(t) = \frac{G(t)}{(b-t)^{1/2}\,(t-a)^{1/2}}\,, \quad h(t) = \frac{H(t)}{(b-t)^{1/2}\,(t-a)^{1/2}}\,,$$

$$p(r) = \frac{F(r)}{(d-r)^{1/2}\, r^{1/2}} \tag{13 a-c}$$

where G, H and F are bounded functions. The solution of (9-11) subject to (12 a-c) may easily be obtained by using the numerical method described in (5).

After determining the density functions g and h the stress intensity factors at the crack tip may be defined and evaluated as follows:

$$k_1(a) = \lim_{x \to a} \sqrt{2(a-x)}\, \sigma_{yy}(x, 0) = \frac{2\mu}{\kappa + 1} \lim_{x \to a} \sqrt{2(x-a)}\, g(x) \tag{14 a}$$

$$k_1(b) = \lim_{x \to b} \sqrt{2(x-b)}\, \sigma_{yy}(x, 0) = -\frac{2\mu}{\kappa + 1} \lim_{x \to b} \sqrt{2(b-x)}\, g(x) \tag{14 b}$$

$$k_2(a) = \lim_{x \to a} \sqrt{2(a-x)}\, \sigma_{xy}(x, 0) = \frac{2\mu}{\kappa + 1} \lim_{x \to a} \sqrt{2(x-a)}\, h(x) \tag{14 c}$$

$$k_2(b) = \lim_{x \to b} \sqrt{2(x-b)}\, \sigma_{xy}(x, 0) = -\frac{2\mu}{\kappa + 1} \lim_{x \to b} \sqrt{2(b-x)}\, h(x) \tag{14 d}$$

Similarly following the procedure given in (5) the mode I stress intensity factors at the inclusion tip may be defined as follows

$$k_1(0) = \frac{\kappa - 1}{2(\kappa + 1)} \lim_{r \to 0} \sqrt{2(-r)}\, p(r)\,, \tag{15 a}$$

$$k_1(d) = -\frac{\kappa - 1}{2(\kappa + 1)} \lim_{r \to d} \sqrt{2(r-d)}\, p(r)\,, \tag{15 b}$$

RESULTS AND DISCUSSION

The main interest of this study is to evaluate the stress intensity factors at the crack and inclusion tips in an infinite strip for various configurations of crack and inclusion for different values of the stiffness parameter γ ($\gamma = \mu\,(1 + \kappa_s) / (A_s\,\mu_s\,(1 + \kappa))$) . These values of the stress intensity factors are given in Figure 2-5. One should also note that the stress intensity factors given in figures are normalized as follows:

$$k_i'(s) = \frac{k_i(s)}{\sigma_o\sqrt{1/2}}\,, \qquad (s = a, b) \tag{16 a}$$

for the crack, and

$$k_1'(t) = k_1(t) / k_o\,, \quad (t = 0, d)\,, \quad k_o = -\frac{\kappa - 1}{2\,\kappa + 1}\,\sigma_o\sqrt{d/2} \tag{16 b}$$

for the inclusion.

From the analysis of Figs. 2-4 it can be seen that the first mode stress intensity factor at the crack tip near the inclusion and second mode stress intensity factors at both end of the crack decrease as the stiffness of the inclusion increases. The first mode stress intensity factors at the inclusion at the and other end of the crack increase as the stiffness of the inclusion increases.

These figures also show that the normalized stress intensity factors k_i', k_2' are strongly dependent on d/l ratio and they all increase with the increase of d/l ratio.

For the cases shown in Figs. 5 due to symmetry of the problems the second mode of stress intensity factors are zero and the first mode stress intensity factors at the crack tips are independent of γ. Whereas the stress intensity factors at both ends of the inclusion decrease as γ increases.

REFERENCE

(1) Erdoğan, F. and Gupta, G.D., Int. J. of Fracture, Vol. 11, 1975. pp. 13-27.

(2) Erdoğan, F. and Gupta, G.D. and Ratwani, M., J. Appl. Mech. Trans. ASME, Vol. 41, 1974, pp. 1007-1013.

(3) Xue-Hui, L. and Erdoğan, F., Eng. Fracture Mech., Vol. 23, No. 5, 1986, pp. 821-832.

(4) Atkinson, C., Int. J. Engng. Sci., Vol 10, 1972, pp. 127-136.

(5) Şahin, A., "The Crack-Inclusion Interaction Problem in an Infinite Strip", M.Sc. Thesis, Middle East Technical University, Ankara, Turkey 1989.

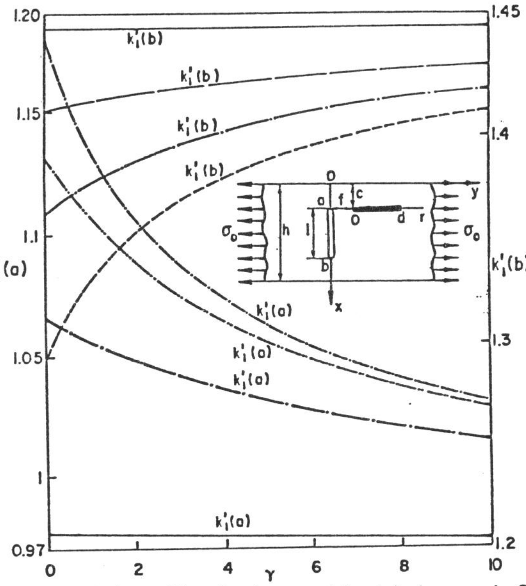

Fig. 2 Stress intensity factor ratio k_1' in an infinite strip containing a crack and an inclusion; $\nu = 0.3$.

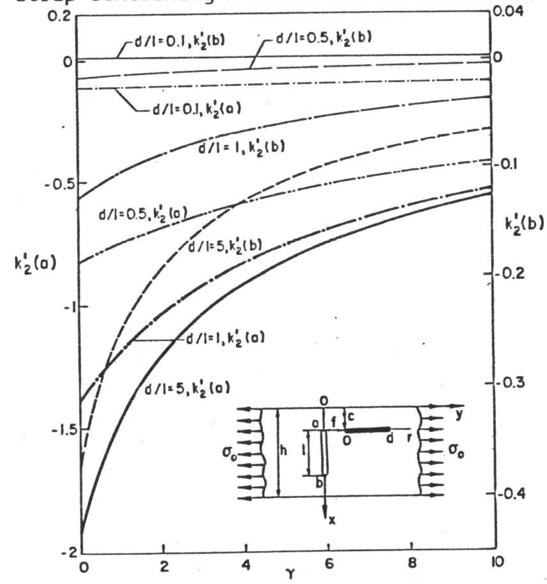

Fig. 3 Stress intensity factor ratio k_2' in an infinite strip containing a crack and an inclusion; $\nu = 0.3$.

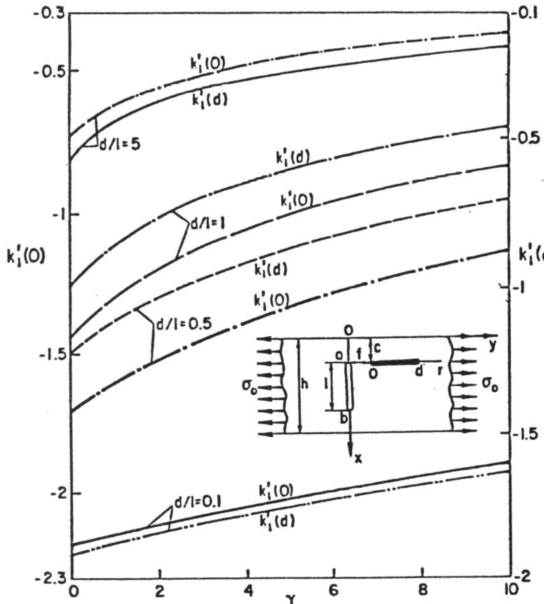

Fig. 4 Stress intensity factor ratio k_1' in an infinite strip containing a crack and an inclusion; ν = 0.3.

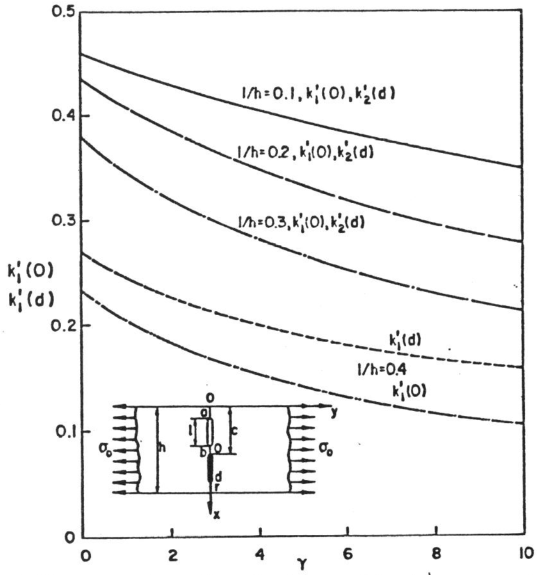

Fig. 5 Stress intensity factor ratio k_1' in an infinite strip containing a crack and an inclusion; ν = 0.3.