THE STRESS APPROACH TO THE CRITERIA FOR MIXED-MODE FRACTURE

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The possibility to determine the critical load and the corresponding direction of propagation under mixed-mode conditions by studying the variation of appropriate stress components at the vicinity of the crack tip forms the contents of the stress approach to mixed-mode fracture(MMF). It consists of the original maxσ_ϑ and the recently proposed maxσ_r, max$|\tau_{r\vartheta}|$, maxσ_1 and maxCOR criteria. A unified presentation of them, with reference to their common underlying principles, is given. The reasons for inclusion of the constant term of stresses in the relevant analysis and the consequences of this inclusion are examined.

INTRODUCTION

The passing from mode-I to mixed-mode results in rotation of the stress field at the crack tip and loss of its symmetry (see e.g. distribution of σ_r in Fig.1).Consequently the crack propagates in a non-self-similar manner under a critical load σ_{cr} different from the σ^I_{cr} of mode I. Determination of direction ϑ_p of propagation and of $\sigma_{cr}/\sigma^I_{cr}$ are the objectives of the criteria of MMF. When dealing with the problem within LEFM it must be:

(i) $r>r_0$ so that all the stresses remain elastic and

(ii) $r<r_1$ so that the stress components may be represented by their singular terms only, in the form:

$$\sigma_{ij}=\frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}f_{ij}(k_1,k_2;\vartheta) = \frac{\sigma}{\sqrt{2r/a}}f_{ij}(k_1,k_2;\vartheta) \qquad (1)$$

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and the strain energy components in the form:

$$\frac{dW}{dV} = \frac{\sigma^2}{r/a} F(k_1^2, k_2^2; \vartheta) \quad (2)$$

Consequently the analysis is developed in the ring R_s (Fig.2b) (see e.g. Irwin**(1)**, Chrysakis**(2)**). In (1),(2) the stress intensity factors K_1, K_2 have been introduced in the form:

$$K_i = \sigma\sqrt{\pi a}\, k_i \ , \quad i = 1,2 \quad (3)$$

where a is a characteristic crack length and the dimensionless factors k_i are functions of the geometry and of the distribution of loading. E.g. in the case of the inclined crack in uniaxial tension (Fig. 2a) it is:

$$k_1 = \sin^2\beta \qquad k_2 = \sin\beta \cos\beta \ . \quad (4)$$

Consider now the problem of determining ϑ_p. All the existing criteria postulate that fracture is related to the position(s) where a critical for the mechanism of fracture quantity obtains its extreme value(s). The various criteria differ in the choice of this quantity, which is either a stress component (Erdogan and Sih**(3)**, Chrysakis **(2,4,5,6)**) or an energy component (Sih**(7)**, Theocaris and Andrianopoulos**(8)**) - hence one may talk about the stress approach or the energy approach to MMF. Now since the algebraic expressions of these quantities (eqs.(1),(2)) are in the form of separable variables, their above mentioned extrema lie on the corresponding surfaces either at the top of hills or at the bottom of valeys, in either case radially arranged along AB (Figs. 2b and 3). And the basic assumption of LEFM, underlying all the criteria, is that this radial direction AB may be extended into the "unknown" plastic circle $C_p(0, r_0)$ up to the crack tip - an assumption justified among othes, by the fact that these criteria were initially proposed for brittle fracture, hence $r_0 \to 0$. The above explain the first of the two hypotheses made by Erdogan and Sih**(3)** in their original criterion: "(a) The crack extension starts at its tip in radial direction" - which is common hypothesis to all the other criteria, too.

Later Williams and Ewing**(9)**, trying to explain a certain discrepancy of the theoretical predictions of the $\max\sigma_\vartheta$ criterion for ϑ_p from the data of their extensive experimental research, proposed the inclusion of the constant term in the expression of σ_ϑ, maximized with respect to ϑ. Thus the expression of σ_ϑ, instead of (1), takes the form:

$$\sigma_\vartheta = \frac{\sigma}{\sqrt{2r/a}} f_\vartheta(k_1,k_2;\vartheta) + \sigma g_\vartheta(\vartheta) \; . \qquad (5)$$

In this case the characteristic of separable variables, encountered in (1),(2), has been lost, hence the value ϑ_p of ϑ maximizing σ_ϑ, being a root of the equation:

$$\frac{1}{\sqrt{2r/a}} f'_\vartheta(k_1,k_2;\vartheta) + g'_\vartheta(\vartheta) = 0 \; , \qquad (6)$$

depends not only on the stress intensity coefficients k_1, k_2 but also on the value of (r/a) (Fig.2b). This type of analysis is necessitated if $r>r_1$ and, for its validity, it should remain $r<r_2$, hence its corresponding region is the ring R_c (Fig.2b). In this region and since, as already explained, $\vartheta_p = \vartheta_p(r/a)$, the locus BC of ϑ_p deviates from the radial direction OAB, hence questions arise, concerning the validity of the basic hypothesis of extension in radial direction. Williams and Ewing report in **(9)** (as corrected by Finnie and Saith**(10)** for the ommision of one term in eq. (3) of ref.**(9)**) improved coincidence with their experimental data by applying eq.(6) with $\sqrt{2r/a}=0.1$. It should be mentioned here aparallel, more general research by Eftis, Subramonian and Liebowitz(see e.g.**(11)**).

The stress approach was identified to the maxσ_ϑ criterion **(3)** until, recently, Chrysakis investigated the contribution of other singular stress components to MMF and proposed the maxσ_r, max$|\tau_{r\vartheta}|$ **(2)**, maxσ_1 **(4)** and maxCOR **(5,6)** criteria. A unified presentation of these stress criteria will be given in the following sections against the background of their common ideas which have been visualized in this Introduction.

THE maxσ_ϑ, maxσ_r, max$|\tau_{r\vartheta}|$ AND maxσ_1 CRITERIA FOR ϑ_p

The following notation is introduced:

$$\left.\begin{array}{llll} s = \sin\frac{\vartheta}{2} & c = \cos\frac{\vartheta}{2} & t = \tan\frac{\vartheta}{2} & \\ s_n = \sin\frac{n\vartheta}{2} & c_n = \cos\frac{n\vartheta}{2} & t_n = \tan\frac{n\vartheta}{2}, & n\ 2,3\ldots \end{array}\right\} \qquad (7)$$

Accordingly the singular terms of the polar components of stresses are written:

$$\sigma_r = \frac{\sigma}{\sqrt{2r/a}}[k_1 c(1+s^2) + k_2 s(1-3s^2)] \qquad (8)$$

$$\sigma_\vartheta = \frac{\sigma}{\sqrt{2r/a}}(k_1 c^3 - 3k_2 sc^2) \qquad (9)$$

$$\tau_{r\vartheta} = \frac{\sigma}{\sqrt{2r/a}}[k_1 sc^2 + k_2 c(1-3s^2)] \qquad (10)$$

Let OP be the radial direction of expected crack propagation and C(O,r) a small circle whose circumference lies in the ring R_s (Fig.3). The separation of the material along OP is connected with tensile forces acting in a direction perpendicular to OP. The most obvious choice of such forces are those corresponding to σ_ϑ. Hence the material will split at that point B of the circumference C (Fig.3), where σ_ϑ becomes maximum **(3)**. The direction ϑ_p of propagation is the root of the equation:

$$\frac{d\sigma_\vartheta}{d\vartheta} = 0 \text{ or } k_1 sc^2+k_2 c(1-3s^2) = 0 \text{ with } \frac{d^2\sigma_\vartheta}{d\vartheta^2}<0 \qquad (11)$$

From (10),(11) it is seen that in case of maxσ_ϑ it is $\tau_{r\vartheta}=0$.

Now the other two components $\sigma_r, \tau_{r\vartheta}$, being also singular, possess high values of the same order of magnitude as σ_ϑ along C, hence they may be expected to contribute to crack propagation as much as σ_ϑ does. To investigate how this may be happening, Chrysakis**(2)** studied the variation of σ_r, $\tau_{r\vartheta}$ along C. To make concrete the discussion, the configuration of the uniaxially loaded inclined crack with $\beta=30^o$ is considered here. The variation of σ_r, σ_ϑ, $\tau_{r\vartheta}$ with ϑ is shown in Fig.4. The maxσ_ϑ occurs for $\vartheta_{p\vartheta}=-60.2^o$. On the other hand σ_r presents two maxima at $\vartheta_{r1}=-163.7^o$ and $\vartheta_{r2}=41.6^o$. The variation of σ_r with ϑ is also shown in Fig.1b in a polar diagram, while in Fig.3 the particular stresses $\sigma_{r1}=\sigma_r(\vartheta=-163.7)$ and $\sigma_{r2}=\sigma_r(41.6)$ are shown applied at the points E_1 and E_2 respectively, symmetrical to the direction of propagation and exerting an "opening action" on the material enclosed in C. Hence the direction of propagation coincides with the bisector of the angle E_1OE_2:

$$\vartheta_r = (\vartheta_{r1}+\vartheta_{r2})/2 = (-163.7+41.6)/2 = -61.0 \simeq \vartheta_{p\vartheta} \qquad (12)$$

This result is valid for all values of β (i.e. of the ratio k_1/k_2) hence it can be postulated that:

"The crack propagates in the direction OP bisecting the angle of directions OE_1, OE_2, where σ_r takes its maximum values."

A similar situation is encountered with max$|\tau_{r\vartheta}|$: there are two maxima at F_1, F_2 with corresponding $\vartheta_{r\vartheta 1}=-117.6^o$ and $\vartheta_{r\vartheta 2}=9.3^o$ and bisector at $\vartheta_{r\vartheta}=-54.1^o$ (Fig.3). At F_1 the shearing stresses are shown to be equivalent to a tension in the direction $F_1'F_1''$ and at F_2 to a tension in the direction $F_2'F_2''$. Hence again the symmetrical positioning of the two maxima of $|\tau_{r\vartheta}|$ with respect to the direction

ϑ_p of propagation may lead to a similar postulate:
"The crack propagates in the direction OP bisecting the angle of OF_1, OF_2 at which $|\tau_{r\vartheta}|$ obtains its maxima."

A third approach to this method of determination of ϑ_p as the bisector of an appropriate angle can be based on the maximization of principal stresses **(4)**. Let σ_1 denote the largest of the two principal stresses. σ_1 is expressed as a function of (r,ϑ) in terms of either $\sigma_r, \tau_{r\vartheta}, \sigma_\vartheta$ given by eqs. (8,9,10) or in terms of $\sigma_x, \sigma_y, \tau_{xy}$ given by similar expressions. The directions ϑ_i of $\max\sigma_1$ are roots of:

$$\frac{\sigma'_x+\sigma'_y}{2}+\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(\frac{\sigma'_x-\sigma'_y}{2}\right)+\tau_{xy}\tau'_{xy}\right]\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau^2_{xy}\right]^{-1/2}=0 \quad (13)$$

where the prime denotes differentiation with respect to ϑ. Eq.(13) has been solved in **(4)** and for $\beta=30^o$ gives $\vartheta_1=-149.4^o$, $\vartheta_2=-14.6^o$. At the corresponding points G_1, G_2 of C the principal stress σ_1 is at an angle $\varphi_i, i=1,2$ to the x-axis, given by:

$$\tan 2\varphi_i=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}=\frac{k_1(-s+s_5)+k_2(3c+c_5)}{k_1(-c+c_5)-k_2(3s+s_5)} \quad . \quad (14)$$

Thus substitution of $\vartheta_1=-149.4$ in (14) gives $\varphi_1=15.8$ for the direction $G_1G'_1$ of σ_1 and $\vartheta_2=-14.6$ in (14) gives $\varphi_2=38.0$ for the direction $G_2G'_2$ of σ_1. The bisector of $G_1G'_1$: $G_2G'_2$ (Fig.5) is in the direction:

$$\vartheta_{\sigma 1} = (\varphi_1-180^o+\varphi_2)/2 \quad (15)$$

and for the above values of φ_1, φ_2 is $\vartheta_{\sigma 1}=-63.1^o$ -which is in good agreement with the direction ϑ_p determined by the other criteria. Thus it may be postulated that:
"When the material enclosed in C is "pulled" by $\max\sigma_1$ at the points G_1, G_2 along the directions $G_1G'_1$, $G_2G'_2$, it is separated along the radial direction OP, parallel to the bisector of $G_1G'_1$, $G_2G'_2$ (Fig.5)."

Numerical values of the variation of ϑ_p ($=\vartheta_r$ or $\vartheta_{r\vartheta}$ or $\vartheta_{\sigma 1}$) with $\beta=\tan^{-1}(K_1/K_2)$ are given in **(2,4)** and its graph in Fig.6 .

DETERMINATION OF LOAD AT FRACTURE $\sigma_{cr}/\sigma^I_{cr}$

The previous discussion for determination of ϑ_p has established two approaches.
(i) That of $\max\sigma_\vartheta$ criterion, in which the σ_ϑ component causes separation of the material of the corresponding surface element $(dr, rd\vartheta)$ at B (Fig.3), hence separation along radius OB is due to the resultant of stresses along OB. If B corresponds to the direction ϑ_p, then the cor-

presponding $\tau_{r\vartheta}=0$, hence the above mentioned resultant per unit thickness, due to σ_ϑ only, is:

$$R_{\vartheta p}= \int_0^r \sigma_\vartheta(r,\vartheta_p)\,dr = 2r\sigma_\vartheta(r,\vartheta_p) \quad . \tag{16}$$

Obviously this resultant is greater than the resultant R_ϑ along any radius $OE(\vartheta)$ other than the direction $OB(\vartheta_p)$ of propagation. Now, since σ_ϑ and σ are proportional, the load σ may be increased until σ_ϑ reaches a critical value $\sigma_{\vartheta,cr}$, at which the material at B , under the sole action of $\sigma_{\vartheta,cr}$ fails, hence $\sigma_{\vartheta,cr}$ is a material constant, hence it is the same to $\sigma^I_{\vartheta,cr}$ under mode I:

$$\sigma_{\vartheta,cr} = \sigma^I_{\vartheta,cr} \tag{17}$$

or, substituting from (9) into the left-hand side of (17) with $\vartheta=\vartheta_p$, $\sigma=\sigma_{cr}$ and into its right-hand side with $\vartheta=0$, $k_1=1$, $k_2=0$, $\sigma = \sigma^I_{cr}$:

$$\frac{\sigma_{cr}}{\sigma^I_{cr}} = \left[k_1\cos^3\left(\frac{\vartheta_p}{2}\right) - 3k_2\sin\left(\frac{\vartheta_p}{2}\right)\cos^2\left(\frac{\vartheta_p}{2}\right)\right]^{-1} \tag{18}$$

where σ_{cr} and σ^I_{cr} stand for the critical loads under mixed mode and mode I conditions respectively.

(ii) In the case of the $\max\sigma_r$, $\max|\tau_{r\vartheta}|$ and $\max\sigma_1$ criteria the bisector approach is followed. In this, it is not the material enclosed in the elementary area $(dr, rd\vartheta)$ which is being pulled at certain point B by the corresponding appropriate normal stresses up to failure; it is the material enclosed in a small circle C which, being pulled by boundary stresses on the arcs AEB, A'E'B' fails along OB (Fig.7). These boundary stresses are σ_r, $\tau_{r\vartheta}$, their resultants R,R' along arcs AEB, A'E'B' respectively are in equilibrium and the projections $R_o=R'_o$ of these resultants on the "opening direction" $DD'\perp OB$ are the ones causing separation along OB. Thus, after determining the ϑ_p of OB by either of $\max\sigma_r$, $\max|\tau_{r\vartheta}|$ and $\max\sigma_1$, the R_o can be determined by the projections of $\sigma_r, \tau_{r\vartheta}$ on DO integrated along the arc AEB **(4,5)**. But if one considers the equilibrium of sector OAEB (and since the crack lips OA,OA' are unloaded) it must be (Fig.8):

$$R_o = R_{\vartheta p} \ , \qquad R_s = R_{r\vartheta p} \tag{19}$$

and, since the direction of propagation OB is approximately the same with that of $\max\sigma_\vartheta$, $R_{\vartheta p}$ is also approximately equal to that evaluated by (16) (and $R_s \simeq 0$). Therefore determination of load at fracture may be reduced to eq.(18) for maxσ_ϑ. These ideas of equivalence in the determination of $(\sigma_{cr}/\sigma^I_{cr})$ may be traced in ref. **(6)**, in which, it has been shown that another stress criterion, the maxCOR (for **C**ircumferential **O**pening **R**esultants) **(5)** is equivalent to the maxσ_ϑ.

* * *

Recapitulating the above discussion:

Certain stress components (namely σ_r, $|\tau_{r\vartheta}|$, σ_1) are distributed in such a way along the arcs AEB, A'E'B', that their maxima appear at points symmetrically located with respect to the direction of propagation (σ_r at E_1 and E_2, $|\tau_{r\vartheta}|$ at F_1 and F_2 and σ_1 at G_1 and G_2, Figs.3,5). Although separation along OB is not due to the sole action of these individual stresses but to the action of the opening resultants R_o, R'_o, still this property of symmetry makes maxσ_r, max$|\tau_{r\vartheta}|$ and maxσ_1 appropriate as criteria for the determination of ϑ_p. On the other hand determination of $\sigma_{cr}/\sigma^I_{cr}$ is algebraically equivalent to that by maxσ_ϑ.

THE INFLUENCE OF THE CONSTANT TERM ON ϑ_p AND $\sigma_{cr}/\sigma^I_{cr}$

The idea behind this modification has been visualized in the present Introduction. If we denote the singular stresses given by (8,9,10) by σ^S_r, σ^S_ϑ, $\tau^S_{r\vartheta}$, then inclusion of the constant term gives for the uniaxially loaded inclined crack (with $s_2=\sin\vartheta$, $c_2=\cos\vartheta$ according to eq.(7)):

$$\sigma_r = \sigma^S_r + \sigma\cos 2\beta\, c_2^2, \quad \sigma_\vartheta = \sigma^S_\vartheta + \sigma\sin 2\beta\, s_2^2, \quad \tau_{r\vartheta} = \tau^S_{r\vartheta} - \sigma\cos 2\beta\, s_2 c_2 \tag{20}$$

In **(9)** eq.(20-b) has been employed to improve the predictions of maxσ_ϑ and in **(12)** eqs.(20) to improve those of maxσ_r, max$|\tau_{r\vartheta}|$, maxσ_1. The direction determined by these improved criteria will be denoted by ϑ^c_p. As alredy explained by eq.(6) in the Introduction, the resulting angles depend on the distance from the crack tip: $\vartheta^c_p = \vartheta^c_p(r/a)$. For $\sqrt{2r/a}=0.1$ the theoretical results reported in both **(9)** and **(12)** fit very well to the experimental data for the brittle PMMA specimens of **(9)**. Still for all the criteria significant variation of ϑ^c_p with r/a has been found. On the other hand, the limits r_1, r_2 of R_c, within which the present analysis is valid, have not been determined and they, as well as the appropriate value of parameter r/a, should depend on the particular material. Thus the basic principle behind the original criteria, that ϑ_p is the direction of the straight line OAB (Fig.2b), since it results for any $r\in(r_0, r_1)$, is replaced here by the following: in these improved criteria ϑ_p^c gives the direction OD, D corresponding to a value of r_p/a which is a material constant. This idea has also been pointed out by Swedlow (13,p.509) in a very interesting critical review of the problems, the methods and the corresponding physical concepts of MMF.

But the inclusion of the constant term in the criteria, in addition to making their predictions ϑ_p^c dependend on the material through r_p/a, it also makes them dependend on the particular configuration of the problem, while in the original (i.e. criteria using singular stresses only) any configuration is reducible to an equivalent - in the sense of having the same ϑ_p - inclined crack configuration. This dependance of ϑ_p^c on the configuration has been pointed out by Ewing et al (14) in a study of the edge crack propagation.

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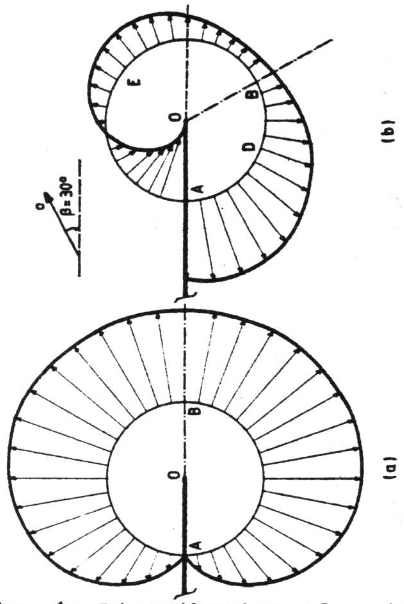

Fig. 1: Distribution of σ_r in mode-I and mixed-mode.

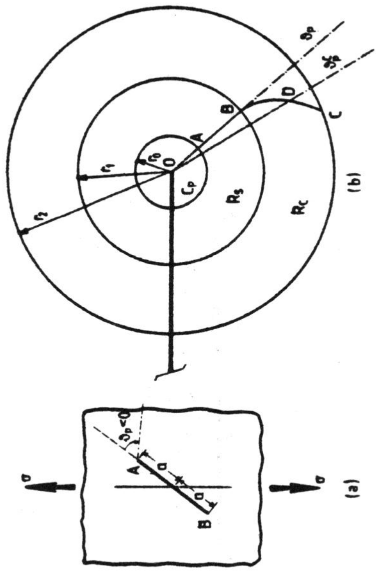

Fig. 2: The inclined crack and the rings at crack tip.

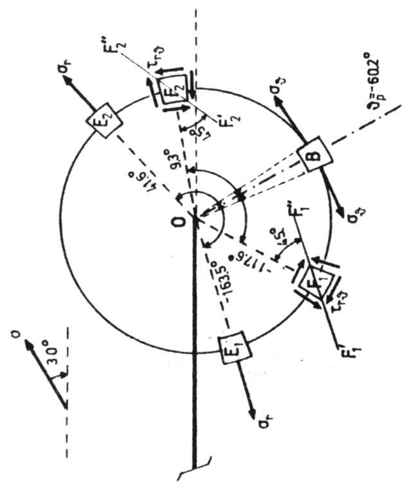

Fig. 3: ϑ_p by maxσ_ϑ, maxσ_r and max$|\tau_{r\vartheta}|$ criteria.

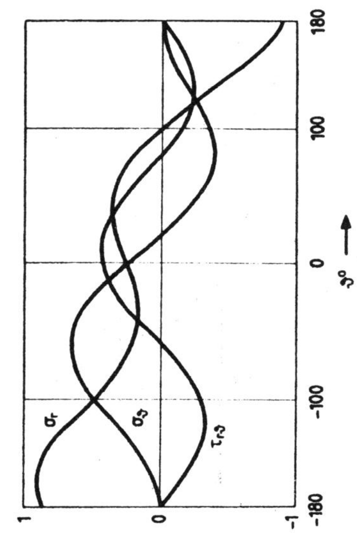

Fig. 4: Variation of the singular stresses.

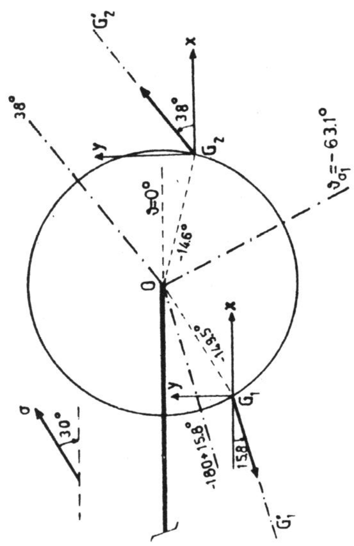

Fig. 5: Determination of ϑ_p by the maxσ_1 criterion.

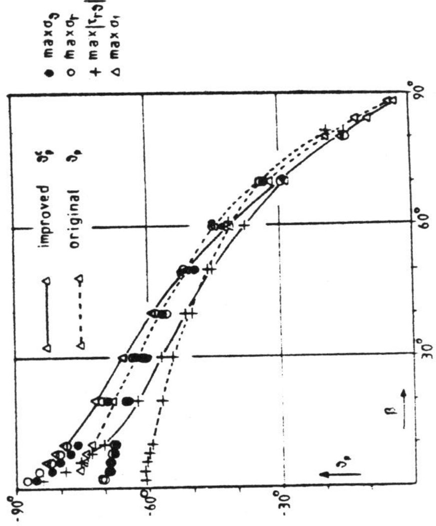

Fig. 6: ϑ_p versus $\beta=k_1/k_2$.

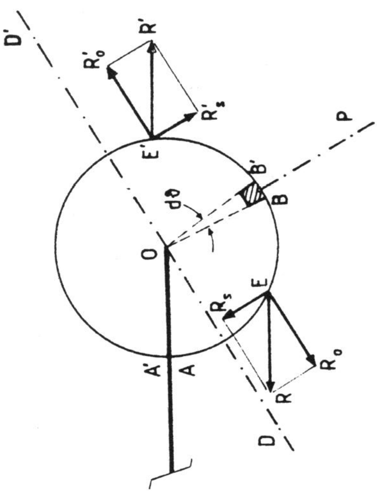

Fig. 7: Stress resultants on arcs AEB and A'E'B'.

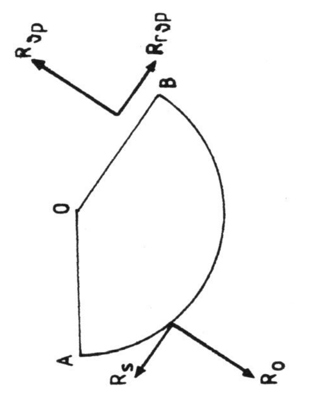

Fig. 8: Stress resultants on sector OAB.