THE BEHAVIOUR OF A PERPENDICULAR PENNY-SHAPED-CRACK SITUATED CLOSELY TO THE FREE SURFACE OF A HALFSPACE

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The comparison of published results for opening mode stress intensity factor solutions for a penny-shaped crack situated closely to the surface of a semi-infinite solid showes a discrepancy between the alternating method solution and the singular integral equation solution. To close the gap a basic solution necessary in the alternating method procedure is extended in a general form.
Using the singular integral equation new results are obtained for cracks positioned very closely to the free surface.

INTRODUCTION

Flaws in pressure vessels, machine components and structural components are often approximated by circular, elliptical or semi-elliptical cracks. When the crack is situated in the neighbourhood of a stressfree surface, the theoretical analysis becomes extremely difficult, since it involves additional geometric parameters describing the dimensions of the elastic solid. The local stresses increase with decreasing distance between the free boundary and the crack (the opposite holds when the crack approaches a rigid boundary). The study of embedded planar cracks near the free surface of a half-space subjected to various loadings has been the subject of special research in the past. A review of literature can be found by Panasyuk et al (1). Solutions for penny-shaped cracks situated parallel to the free surface of a half-space have recently been published by Kuzmin and Ufland (2), Guz and Nazarenko (3) and Srivastava and Singh (4).

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Stress intensity factors for an embedded elliptical crack normal to the boundary in a halfspace approaching the free surface, are also available. Shah/Kobayashi (5) solved this problem using the Schwarz-Neumann alternating method as described by Kantorovich and Krylov (6). The analysis from Nisitani and Murakami (7), Isida and Noguchi (8) has been performed using the body force method. An approximated Mode-I-solution is available from Smith and Alavi (9) using the alternating method too. Kaya (10) solved the same problem by a singular integral equation.

Comparing Alavi's and Kaya's stress-intensity factors for the point on the crackfront nearest to the free surface (Polarangle $\theta = 180°$) for the aspect-ratio $h/a = 1{,}1$ discrepancies of 10% are found. The work reported is an extension and a refinement of the work by Alavi to close the gap. Based on the singular integral equation new results are computed for penny-shaped cracks very close to the free surface.

THE ALTERNATING METHOD - SOLUTION PROCEDURE

The alternating method (11, 12) is an iterative procedure which may be used to solve problems with complicated geometry. Alavi used it for the problem of a circular crack in a semi-infinite solid (Figure 1). The stress-free condition at $x = -h$ may be satisfied by an iterative method of adding various solutions together. The method is explained by the following steps shown in Figure 2.

Step 1. It is assumed that there exists no crack. The normal stresses due to applied load at the location of the crack surfaces are found.

Step 2. Now the existence of a crack is taken into account. The normal stresses found in step 1 must then be removed by applying equal and opposite normal stresses to the crack surfaces. In order to do this, it is necessary to find a solution for a circular crack embedded in an infinite solid subjected to an arbitrary normal loading on the crack surface.

In this paper the authors present a generalisation of Alavi's penny-shaped crack solution to compute the normal and shear stresses on the plane $x = -h$.

Step 3. These residual stresses on the surface plane are brought to vanish by applying opposite surface loadings on the boundary of an uncracked semi-infinite solid. The normal stresses on the crack surface resulting from this removal of stresses from the free plane are then computed. Due to symmetry the shear stresses vanish.

Alavi used for this "freeing-process" basic solutions for a semi-infinite solid when a small rectangular area of its surface is subjected to constant normal and shear stresses. For this step a new half-space-solution for a bilinearely distributed normal and shear stress over a rectangular area has been derivated and is in preparation for publication.

Step 4. By applying opposite stresses, σ^1_{zz} is erased from the crack plane, but this will again cause some residual tractions on the x=-h plane.

Step 5. Steps 3 and 4 are repeated until the residual stresses on the crack plane and on the x = -h plane become negligible.

The final solution is obtained by superposing the results of each iteration step.

PENNY SHAPED CRACK SOLUTION

Consider an elastic infinite solid containing a circular crack of radius a in the plane z = 0 which is opened by a normal pressure p (r, θ) symmetrical with respect to the plane z = 0. Using cylindrical coordinate system (r, θ, z) and expressing the loading term in a Fourier cosine series

$$\left(\frac{1-2\nu}{2}\right)\bar{p}(r,\theta) = \sum_{n=0}^{\infty} B_n(r)\cos(n\theta)$$

the problem reduces to the following dual integral equation for the unknown functions $f_n(\xi)$:

$$\int_0^{\infty} \xi\cdot J_n(\xi r)\cdot f_n(\xi)\cdot e^{-z\xi}\,d\xi = B_n(r) \qquad 0 < r \leqslant 1$$

$$\int_0^{\infty} J_n(\xi r)\cdot f_n(\xi)\cdot e^{-z\xi}\,d\xi = 0 \qquad r > 1$$

Expanding the Fourier coefficients $B_n(r)$ into the power series

$$B_n(r) = \sum_{p=0}^{\infty} C_n^p\cdot r^p$$

the final expression for $f_n(\xi)$ arises

$$f_n(\xi) = \frac{1}{\sqrt{2}}\sum_{p=0}^{\infty} C_n^p H_n^p\cdot \xi^{1/2}\int_0^1 \eta^{p+\frac{3}{2}}\cdot J_{n+\frac{1}{2}}(\xi\eta)\,d\eta \qquad (5)$$

with

$$H_n^p = \frac{\Gamma\left(\frac{p+n+2}{2}\right)}{\Gamma\left(\frac{p+n+3}{2}\right)}\,.$$

For values r and z greater than unity, the series expansion of the Bessel function $J_{n+1/2}(\xi\eta)$ was used. Performing the integration $f_n(\xi)$ results in

$$f_n(\xi) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} A(n,p,k) \cdot C_n^p \cdot \xi^{2k+n+1}$$

where

$$A(n,p,k) = \frac{(-1)^k \cdot H_n^p}{2^{2k+n+1} k!\,(2k+n+p+3) \cdot \Gamma(k+n+\frac{3}{2})} \,.$$

The stress components, e.g. $\tau_{\theta z}$, result now as

$$\tau_{\theta z} = \left(\frac{-2}{1-2\nu}\right) \sum_{n=0}^{\infty} n \frac{z}{r} \sin(n\theta) \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} A(n,p,k) \cdot C_n^p \cdot S(n, 2k+n+2) \,.$$

The integrals

$$S(m,n) = \int_0^{\infty} \xi^n \cdot J_m(\xi r) \cdot e^{-z\xi} d\xi$$

necessary for expressing the stress components were computed by the use of recurrence formula given in (13). For this case the series converge after only few terms.

For values of r and z less than unity, the integral in (5) was evaluated by Alavi in closed form only for values n = p = 0, 1, 2 due to the extreme complexity of including more terms in the series.

GENERALIZED SOLUTION

In (14) an exact solution for the integral in (5) is given using the Lommel functions $S_{m,n}(\xi)$,

$$\int_0^1 \eta^m \cdot J_n(\xi\eta) d\eta = \xi^{-m-1} \cdot \Big[(m+n-1)\xi \cdot J_n(\xi) \cdot S_{m-1,n-1}(\xi) -$$

$$- \xi \cdot J_{n-1}(\xi) \cdot S_{m,n}(\xi) + 2^m \frac{\Gamma(\frac{m+n+1}{2})}{\Gamma(\frac{n-m+1}{2})} \Big] ; \quad \begin{array}{l} \xi > 0. \\ Re(m+n) > -1 \end{array}$$

Applying some recurrence formulas for $S_{m,n}(\xi)$ (a detailed description is given in (15)) the integral (5) can be computed analytically.

SINGULAR INTEGRAL EQUATION SOLUTION

The singular integral equation for a semi-infinite solid with a penny-shaped crack perpendicular to the boundary, located on z=0 plane and occupying a region specified by $(x,y)\in\Omega$ can be expressed as

$$\iint_{\Omega} \frac{W(x_0,y_0)\,dx_0\,dy_0}{\left[(x_0-x)^2+(y_0-y)^2\right]^{3/2}} + \iint_{\Omega} W(x_0,y_0)\cdot K(x_0,y_0;x,y)\,dx_0\,dy_0 = \frac{4\pi(1-\nu)}{\mu}\cdot p(x,y) ,$$

where $K(x_0,y_0;x,y) = \dfrac{1}{\left[(x_0+x+2h)^2+(y_0-y)^2\right]^{3/2}} + F(x_0,y_0;x,y);$

$$F(x_0,y_0;x,y)=6\Big\{(1-2\nu)^2\frac{1}{R(R+x_0+x+2h)^2}+\frac{4\nu}{R^3}\left(\frac{1}{3}-\nu\right)-$$

$$-(1-2\nu)(x_0+x+2h)\left[\frac{1}{R^3(R+x_0+x+2h)}+\frac{}{R^2(R+x_0+x+2h)^2}\right]-$$

$$-(1-2\nu)2\nu\frac{(x_0+x+2h)^2}{R^5}+\frac{3}{R^5}(x_0+h)(x+h)\Big\};$$

$$R=\sqrt{(x_0+x+2h)^2+(y_0-y)^2}$$

The behaviour of the unknown crack opening displacement $W(x_0, y_0)$ near the boundaries of Ω can be represented by a weight function $w(x_0, y_0)$:

$$W(x_0,y_0)=g(x_0,y_0)\cdot w(x_0,y_0)=g(x_0,y_0)\sqrt{a^2-x_0^2-y_0^2} .$$

The new unknown function $g(x_0, y_0)$ is now approximated in terms of a double powers series

$$g(x_0,y_0)=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{ij}\,x_0^i\,y_0^j .$$

To determine the $(N_1+1)(N_2+1)$ unknown coefficients a_{ij} the integral equation is evaluated at certain collocation points $(x_K, y_K)\in\Omega$:

$$x_K = 0{,}96.a.\cos\varphi , \qquad y_K = 0{,}96.a.\sin\varphi ,$$

$$x_K=\sqrt{\frac{6+\sqrt{6}}{10}}\,a.\cos\varphi , \qquad y_K=\sqrt{\frac{6+\sqrt{6}}{10}}\,a.\sin\varphi ,$$

$$x_K=\sqrt{\frac{6-\sqrt{6}}{10}}\,a.\cos\varphi , \qquad y_K=\sqrt{\frac{6-\sqrt{6}}{10}}\,a.\sin\varphi ,$$

where $\varphi = \frac{(2j+1)}{20}\pi, \; j = 0,1,\ldots,9; \quad K = 1,\ldots M; \quad M \geq (N_1+1)(N_2+1).$

Finally the following linear algebraic equations are obtained:

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{ij}\left[C_{ij}(x_K,y_K) + H_{ij}(x_K,y_K)\right] = \frac{4\pi(1-\nu)}{\mu}p(x,y).$$

The two-dimensional singular integrals over a circular region

$$C_{ij}(x_K,y_K) = \iint_{\Omega} \frac{x_o^i y_o^j \sqrt{a^2-x_o^2-y_o^2}}{\left[(x_o-x_K)^2+(y_o-y_K)^2\right]^{3/2}}\,dx_o\,dy_o$$

are computed with the help of a computer program from the closed form expression derived in (10).
The regular integrals

$$H_{ij}(x_K,y_K) = \iint_{\Omega} x_o^i y_o^j \sqrt{a^2-x_o^2-y_o^2}\cdot K(x_o,y_o;x_K,y_K)\,dx_o\,dy_o$$

was solved using the NAG/LIB Subroutine D01JAF. Solving these equations using the method of least squares (NAG/LIB Subroutine F04JGF) the stress intensity factor is calculated from

$$k_1 = \frac{\mu\sqrt{a}}{2(1-\nu)}\cdot g(a\cos\theta, a\sin\theta).$$

Results for constant pressure $p(x,y) = -p_0$, and a Poisson's ratio of $\nu = 0,3$ are shown in Figure 3.

SYMBOLS USED

a	= circular crack radius (mm)
ν	= Poisson's ratio (-)
p	= pressure distribution (N/mm²)
μ	= shearing modulus (N/mm²)
r,z	= cylindrical coordinates, nondimensionalized through division by the crack radius
$\bar{p}$	= pressure distribution, nondimensionalized through division by the shearing modulus
$J_n(\xi)$	= Bessel function of the first kind of order n

$\Gamma(\xi)$ = Gamma function

$S_{m,n}(\xi)$ = Lommel function

Ω = plane crack area (mm^2)

$⨏$ = finite part integral, see for definition (16)

REFERENCES

(1) Panasyuk, V.V., Andrejkiv, A.E., Stadnik, M.M., Engng.Fracture Mech., Vol.14, 1981, pp. 245 - 260.

(2) Kuzmin, I.N., Ufland, Y.S., Prik.Math.Mech., Vol.29, 1965, pp. 1132 - 1137.

(3) Guz, A.N., Nazarenko, V.M., Theor. & Appl. Fracture Mech., Vol.3, 1985, pp. 233 - 245.

(4) Srivastava, K.N., Singh, K., Int.J.Engng.Sci., Vol.7, 1969, pp. 469 - 490.

(5) Shah, R.C., Kobayashi, A.S., Int.J. Fracture, Vol. 9, 1973, pp. 133 - 146.

(6) Kantorovich, L.V., Krylov, V.I., "Approximate Methods of Higher Analysis", Interscience Publishers Inc., New York, 1964.

(7) Nisitani, H., Murakami, Y., Int.J.Fracture, Vol.10, 1974, pp. 353 - 368.

(8) Isida, M., Nogouchi, H., Engng.Fracture Mech., Vol.20, 1984, pp. 387 - 408.

(9) Smith, F.W., Alavi, M.J., Engng.Fracture Mech., Vol.3, 1971, pp. 241 - 254.

(10) Kaya, A.C. "Application of Integral Equations with Strong Singularities in Fracture Mechanics", Ph.D.Thesis, Lehigh University, 1984.

(11) Hartranft, R.J., Sih, G.C. "Alternating Method Applied to Edge and Surface Crack Problems", Technical Report NASA-TR-72-1, Lehigh University, 1972.

(12) Atluri, S.N. "Computational Methods in the Mechanics", Vol.2, Elsevier Science Publishers, Amsterdam, 1986.

(13) Alavi, M.J. "Stress Near a Circular Crack in a Half-space", Ph.D.Thesis, Colorado State University, 1968.

(14) Gradshteyn, I.S., Ryzhik, I.M. "Tables of Integrals, Series and Products", Academic Press, New York, 1980.

(15) Oberhettinger, F. "Tables of Mellin Transforms", Springer Verlag, Berlin Heidelberg New York, 1974.

(16) Hadamard, J. "Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, 1923.

ILLUSTRATIONS

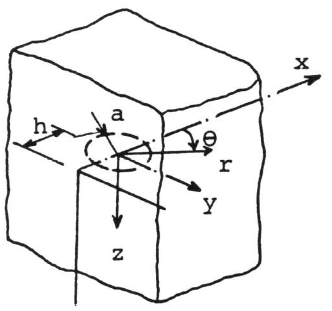

Figure 1. Penny-shaped crack near a free boundary

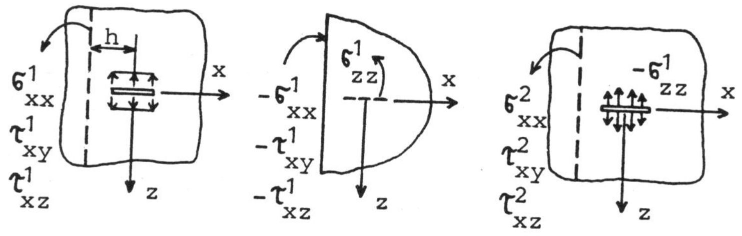

Figure 2. Alternating method

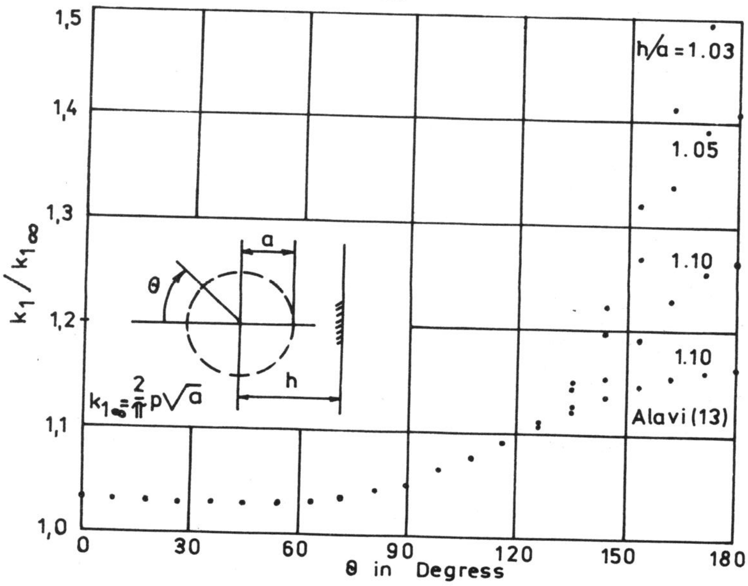

Figure 3. Stress intensity factor for an penny-shaped crack in a halfspace in simple tension