STUDY OF CRACK TRAJECTORIES IN THIN CYLINDRICAL SHELLS: NUMERICAL ANALYSIS AND EXPERIMENTAL VALIDATION

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Crack paths in cylindrical shells have been analysed in order to correlate their geometrical characteristics to the mechanical state. Crack paths predictions have been carried out by using a variational principle that is equivalent to the maximization of the energy released in the body. The Strain Energy Density criterion has been also employed by using the stress-strain fields in the cracked shells, evaluated by Finite Element Method. An experimental investigation by using the Caustics Method has been carried out on plexiglas cracked shells.

INTRODUCTION

The equilibrium and propagation of cracks have been extensively investigated in the last decade. Both the thermodynamical and the mechanical description have been used to study the conditions governing the extension and evolution of cracks.

Rice [1] and Gurtin [2] have studied the thermodynamics of Griffith cracks giving a refined interpretation of the Griffith criterion for quasi-static crack growth. Rice has found some "global restrictions" on the quasi-static brittle fracture. Gurtin has shown that Griffith criterion is a necessary condition for crack initiation in non-linear thermoelasticity, provided that some proper conditions are satisfied.

Fracture criteria are based on both energy balance equation (global condition) and local stress quantities. The characteristics of crack trajectories are correlated to the mechanical state occurring in the neighbourhood of the crack tip and to energy exchange between different zones of the whole structure [3].

Among such fracture criteria, the Maximum-Stress (MS), Minimum-

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Strain-Energy-Density (SED) and Maximum-Energy Release Rate (MG) have been used to predict the crack paths and the critical load at which unstable propagation occurs. Wu [4] has shown that the MG criterion is able also to take into account the coupling phenomena between plane and antiplane loads.

In this paper, cylindrical cracked shells have been studied. Initial curvature gives rise to coupling between in-plane and out-of-plane behaviour, that means the fracture problem must be solved by taking into account both the plane and antiplane loads.

To predict the crack path the MG and the SED-criterion have been employed. The MG criterion is based on the principle of minimum potential energy and by a proper formulation of the corresponding functional a simple procedure for crack paths prediction has been developed.

GOVERNING EQUATIONS AND ELASTIC FIELDS

Cracked shells theory has been developed by several authors and different solution procedures and methods have been proposed, [5]-[7]. Most of mathematical models refer to shallow shells theory by considering a restricted domain containing the crack. Naghdi [8] has proposed a set of equations for shallow shells, where the transvers shear deformation is taken into account and from which the Marguerre's equations can be recovered.

The shallow shell equations can be obtained by projecting the net of coordinates of the middle surface on to a plane and by writing the field equations in this new net of coordinates. Neglecting the terms containing squares and product of z and its derivatives compared to unity the governing differential equations can be obtained. Let us consider the equilibrium equations:

$$N_{ij,j} = 0 \tag{1.a}$$

$$V_{i,i} + [z_{,j} N_{ij}]_{,j} = 0 \tag{1.b}$$

$$V_i = M_{ij,j} \tag{1.c}$$

The stress-strain relations are expressed by:

$$\varepsilon_{ij} = \frac{1}{2} [(u_{i,j} + u_{j,i}) + (z_{,j} w_{,i} + z_{,i} w_{,j})]$$

$$\varepsilon_{ij} = \frac{1}{Eh} [- \nu N_{kk} \delta_{ij} + (1 + \nu) N_{ij}] \tag{2.a}$$

$$M_{ij} = \frac{1}{2} D [2\nu \gamma_{k,k} \delta_{ij} + (1 - \nu) (\gamma_{i,j} + \gamma_{j,i})] \tag{2.b}$$

$$\gamma_i = - w_{,i} \tag{2.c}$$

Introducing the stress function defined by:

$$N_{ij} = (\nabla^2 F)\,\delta_{ij} - F_{,ij} \tag{3}$$

and its substitution into the compatibility equation yields the following fourth-order differential equation, governing the extensional behaviour:

$$\nabla^2(\nabla^2 F) = Eh(2z_{,12}\,w_{,12} - z_{,11}\,w_{,22} - z_{,22}\,w_{,11}) \tag{4}$$

Substitution of the stress-strain relations into the flexural equilibrium equation yields:

$$D\,\nabla^4 w = [\,(\nabla^2 z)\,\nabla^2 F - z_{,ij}\,F_{,ij}\,] \tag{5}$$

We have thus two coupled fourth-order differential equations in terms of the two unknown functions: w and F. For a cylindrical shell one of the radii of curvature is infinite while the other is constant. The coupled differential equations (4) and (5) can be reduced to a set of singular integral equations, whose solution can be obtained analytically, for small values of the shell parameter (which is proportional to the crack lenght-shell radius ratio), or numerically.

In this work, the displacement and stress functions have been expressed in the following form:

$$w = \Sigma_n\,\bar{w}_n\,e^{\alpha_n x}\cos(ny/R) \tag{6.a}$$

$$F = kw \tag{6.b}$$

where k is a proportional factor to be determined.

By substituting the (6) into the (4), (5) the characteristic equations are obtained:

$$\begin{cases} -\dfrac{Eh}{Rk}\,\alpha_n^2 + \alpha_n^4 + (n/R)^4 - 2\alpha_n^2(n/R)^2 = 0 \\ \alpha_n^4 + (n/R)^4 - 2(n/R)^2\,\alpha_n^2 + \dfrac{k}{RD}\,\alpha_n^2 = 0 \end{cases}$$

that, if $k = \pm j\sqrt{EhD}$, can be reduced to the alone equation:

$$\alpha_n^4 + (n/R)^4 - 2(n/R)^2\,\alpha_n^2 + \frac{j}{R}\sqrt{\frac{Eh}{D}}\,\alpha_n^2 = 0 \tag{7}$$

Once the solution of the characteristic equation has been determined the displacement and stress functions are defined, and the stress and displacement fields can be evaluated. For an arbitrarely oriented crack the governing equations are the followings:

$$\nabla^4 F - Eh\left(\frac{1}{R_y}\frac{\partial^2 w}{\partial x^2} + \frac{1}{R_x}\frac{\partial^2 w}{\partial y^2} + \frac{2}{R_m}\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \tag{8.a}$$

$$\nabla^4 w + 1/D\left(\frac{1}{R_y}\frac{\partial^2 F}{\partial x^2} + \frac{1}{R_x}\frac{\partial^2 F}{\partial y^2} + \frac{2}{R_m}\frac{\partial^2 F}{\partial x \partial y}\right) = 0 \qquad (8.b)$$

where:

$$R_x = R/\sin^2\alpha \qquad R_y = R/\cos^2\alpha \qquad R_m = \sqrt{R_x R_y} = R/(\sin\alpha\,\cos\alpha)$$

If the displacement and stress functions are expressed as:

$$w = \Sigma_n\, \bar{w}_n\, e^{\alpha_n x}\, e^{j g_n y} \quad \text{and} \quad F = kw \qquad (9)$$

the characteristic equation is:

$$\alpha_n^4 - 2\alpha_n^2 g_n^2 + g_n^4 + jS\left(\frac{\alpha_n^2}{R_y} - \frac{g_n^2}{R_x} + 2\frac{j\,\alpha_n g_n}{R_m}\right) = 0 \qquad (10)$$

where $S = \sqrt{Eh/D}$. The corresponding roots have been evaluated numerically (once the g_n have been prescribed). To the perturbation solution, we must add the solution of the uncracked shell, with uniform stresses distribution.

CRACK TRAJECTORIES PREDICTION AND FRACTURE CRITERION

Let us consider a net of coordinate curves (ξ_1, ξ_2) on the middle surface of a shell. By a parallel projection on to a plane π, new coordinate curves are obtained and a continuous and one-to-one correspondence between points of the surface S and of the plane π is established, Fig. 1.

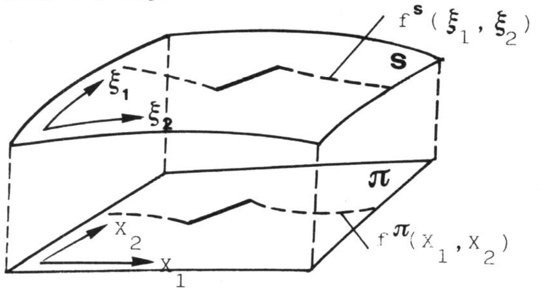

Fig. 1 - Element of middle surface S and its projection on a plane π.

We are concerned here with crack trajectories on the middle surface of shells, which can be described by a certain function $f^S(\xi_1, \xi_2)$ where ξ_i are the curvilinear coordinates on S.

By virtue of the correspondence between ξ_1 and x_i, the function describing the crack trajectory can be also expressed in terms of the projected coordinates x_i, i.e. $f^\pi(x_1, x_2)$.

By introducing a projection operator Π, and its invers Π^{-1} the following transformations hold:

$$\Pi[f^S(\xi_1,\xi_2)] = f^\pi(x_1,x_2)$$
$$\Pi^{-1}[f^\pi(x_1,x_2)] = f^S(\xi_1,\xi_2) \qquad (11)$$

To find the projected crack trajectory we follow the method described in [9], where a variational condition is used. We outline briefly here the basic concepts of the method. Assuming a perfectly elastic body, the Griffith fracture criterion can be expressed in the following form:

$$\delta \int_S [2\Gamma - p_i u_i] \, ds = 0 \tag{12}$$

where $p_i = -\sigma_{ij} n_i$ are the stress components acting on planes coinciding with the crack surface, σ_{ij} are the stresses occurring in the uncracked body, n_i are the component of the outward normal to the crack surface, u_i are the displacement components of the points of the crack surface for the cracked body loaded on the crack surface by a pressure p_i.

To apply the variational condition (12) the displacements u_i are needed. They can be evaluated by mapping the crack into a circle of unit radius, and the region outside the crack into the outside of the circle. The details of the method are described in [9]. The numerical procedure is based on the approximation of the trajectory with a piecewise curve, consisting of segments characterized by their length l_k and slope $\bar{\theta}_k$ to the x axis. The following equation for the trajectory is obtained:

$$y = x\left(-\sum_{m=0}^{k-1} l_m \cos\bar{\theta}_m\right)\tan\bar{\theta}_k + \sum_{m=0}^{k-1} l_m \sin\bar{\theta}_m \; ;$$

$$\sum_{m=0}^{k-1} l_m \cos\bar{\theta}_m \leq x \leq \sum_{m=0}^{k} l_m \cos\bar{\theta}_m \; ; \qquad k = 0, \pm 1, \pm 2 \ldots \tag{13}$$

Following the same procedure as in [9], the energy functional is expressed as a function of a certain number of variables:

$$F(p_j, \bar{\theta}_0, \bar{\theta}_1 \ldots \bar{\theta}_n, L_n) = \int_S [2\Gamma - p_j u_j (p_j, \bar{\theta}_0, \bar{\theta}_1 \ldots \bar{\theta}_n, L_n)] \, ds \tag{14}$$

where $L_n = \Sigma_{k=0}^{n} l_k$.

The problem is now reduced to that of finding extreme points of $F(p_j, \bar{\theta}_n, L_n)$ that yields the following system of equations:

$$\frac{\partial F}{\partial \bar{\theta}_i} = 0 \qquad i = 1, 2, \ldots n$$

$$\frac{\partial F}{\partial L_n} = 0 \tag{15}$$

The transformation of the projected crack trajectory to the actual one, must be carried out by taking into account the transformation of the metric tensors of the plane π and the middle surface S.

RESULTS AND DISCUSSION

The elastic fields in the neighbourhood of the crack tip have been evaluated for cracked cylindrical shells characterized by small shell parameters. The loading conditions were of uniform traction along the shell axis. The stress resultants and SED distributions have been computed for single and mixed mode crack configurations. Fig. 2.a and 2.b show the extensional stress resultants evaluated along a circle centered at the crack tip for two different polar distances. These distributions sound like those of cracked plate, where a hydrostatic stress field occurs for $\theta = 0$ at small distance from the tip. At higher distances from the tip, the stresses are no longer hydrostatic. The SED, reported in Fig. 2.c, has a maximum for $\theta = 0$ and its distribution is quite dissimilar from that of cracked plate, where the maximum occurs at $\theta = 70°$. Fig. 3.a shows the distributions of the same quantities around the tip of an inclined crack. In Fig. 3.b the principal stress resultants are reported. The maximum values of N_1 and N_2 occur at $\theta \cong -30°$ and $\theta \cong -20°$ respectively while the maximum SED value occurs at $\theta \cong -20°$.

Elastic stress fields and SED distributions have been also evaluated by FEM. The computations have been carried out by using a finite elements code based on the "p-convergence" theory of FEM and in which "hierarchic elements" are used.

Three-dimensional elements (hexahedron and pentahedron) have been employed and Triangular-Quarter-Point (TQP) elements have been used to model the stresses singularity at the crack tip. The energy release rate have been evaluated by the Virtual Crack Extension (VCE) method. Fig. 4 shows the stresses and SED distributions around the tip of a circumferencial crack. In Fig. 5 the SED distribution for a mixed mode configuration is reported. The maximum $(SED)_{min}$ occurs at $\theta \cong -38°$. Fig. 7 and 8 show the SED distributions, for a circumferencial and inclined crack, evaluated by using the stress intensity factors values computed by the Caustic Method [10]. It can be seen that the stresses and SED distributions evaluated by using the closed solution can be effectively used only for a qualitative analysis. Of course quantitative computations cannot be carried out because singular eigenfunctions are necessary to model the singularity in the elastic fields at the crack tip. The elastic fields evaluated by the FEM show that at the crack tip a 3D stress field occurs.

The variational method, that is strictly correlated to the MG criterion allows to take into account the redistributions of the stress fields with crack growth. This characteristic is particularly useful for elasto-plastic fracture problem.

CONCLUDING REMARKS

Application of fracture criteria for the prediction of quasi-static unstable crack propagation, is an important step in failure analysis of structural components. In this paper a simple method for the prediction of crack trajectory, in shallow shells, has been proposed. Its application has shown that it provides satisfactory results for initial stages of crack propagation, where the assumption of shallowness can be considered to be valid.

When the crack lenght increases to an amount such that the shallow shells theory cannot be applied, the proposed method fails to give good results. This can be explained because:

- the elastic fields are affected by some errors deriving from the lack of correctness of the governing equations;
- the employed projection of a restricted domain of the middle surface is no longer applicable.

The proposed variational condition, as fracture criterion, is based on a general physical principle, that is the minimum of the total potential energy. It follows that the fracture criterion is correlated to overall conditions in the structure, and it cannot take into account microstructural parameters describing microscopic phenomena. Nevertheless, the variational condition is derived from the Griffith criterion, which has been interpreted and widely analyzed from the thermodynamical point of view.

About the elastic fields evaluated by using the series expansions of the displacement and stress functions, the following remarks can be done:

- satisfactory results can be obtained by using high order terms in the series expansion;
- asymptotic computation are necessary in order to obtain satisfactory qualitative results;
- the proposed solutions don't lend for quantitative computation.

The numerical results obtained by the FEM have shown a good agreement with the expected distributions. The SED criterion, used on the basis of the elastic fields evaluated by the FEM, has provided accurate results for the prediction of fracture angles. It has been confirmed that it's able to take into account of three-dimensional fields that occurs at the crack tip.

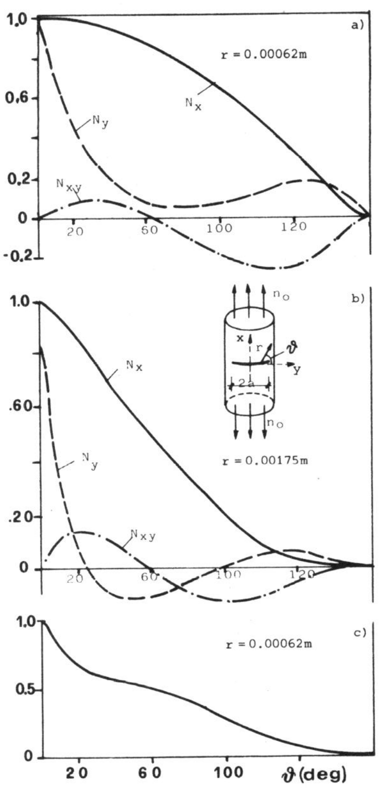

Fig. 2 - Normalized stress resultants (a,b) and normalized SED (c) distributions in the neighbourhood of the crack tip for single mode configuration (R = 0.015 m, 2a = 10^{-2} m).

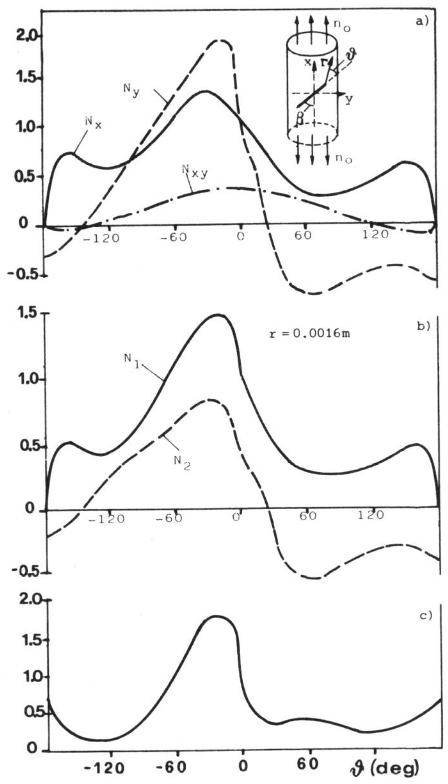

Fig. 3 - Normalized stress resultants (a), normalized principal stress resultants (b) and normalized SED (c) distributions for mixed mode configuration (R = 0.015 m, 2a = 10^{-2} m, β = 60°).

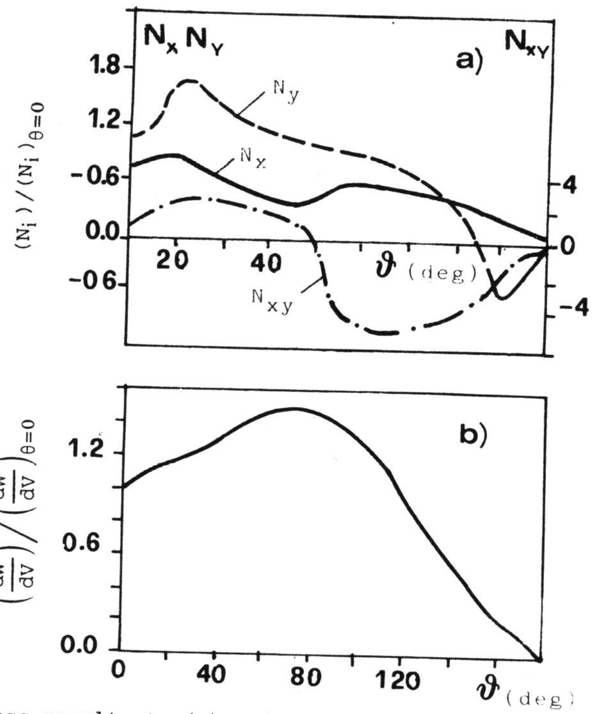

Fig. 4 - Stress resultants (a) and SED (b) distributions around the tip of a circumferencial crack, evaluated by FEM (R = 0.015 m, 2a = 10^{-2} m, $\beta = 90°$).

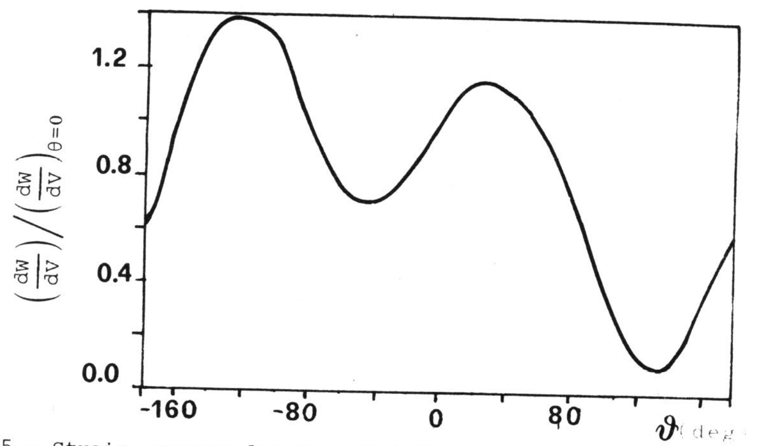

Fig. 5 - Strain energy density distribution around the tip of an inclined crack evaluated by FEM (R = 0.015 m, 2a = 10^{-2} m, $\beta = 60°$).

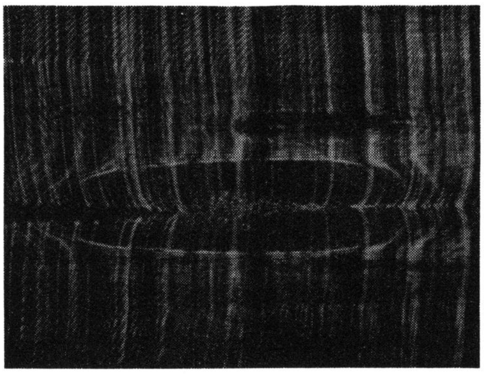

Fig. 6 - Experimental reflected caustic from a circumferencially cracked shell ($R = 0.025$ m, $2a = 10^{-2}$ m, $\beta = 0$).

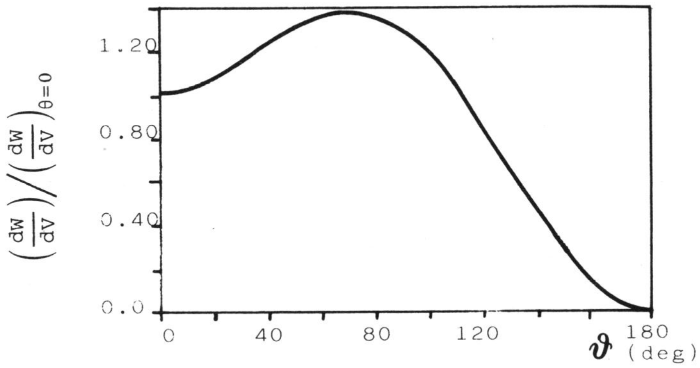

Fig. 7 - Strain energy density distribution, evaluated by the Caustic Method ($R = 0.025$ m, $2a = 10^{-2}$ m, $\beta = 0°$).

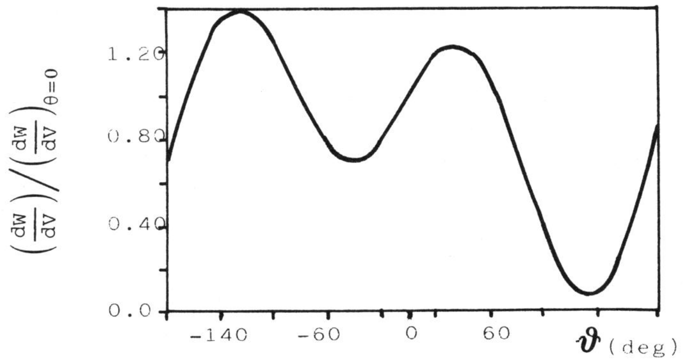

Fig. 8 - Strain energy density distribution evaluated by the Caustic Method ($R = 0.025$ m, $2a = 10^{-2}$ m, $\beta = 30°$).

SYMBOLS USED

a = half crack lenght
h = shell thickness
r = polar coordinate
u,v = displacement on middle surface
w = displacement in the z-direction
F = stress function
E = Young modulus
N = stress resultant
V = transvers stress resultant
M = stress couple
R = shell radius
β = loading angle
α = crack angle
ν = Poisson modulus
θ = polar angle
Γ = surface fracture energy density.

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