# ON THE RESEARCH OF MICROSTRESSES IN THE NEIGHBOURHOOD OF A CRACK IN COMPOSITE MATERIALS

V.Z. Parton*, A.L. Kalamkarov*, B.A. Kudryavtsev*

We consider the problem of a straight-line crack in the highly heterogeneous (composite) elastic medium with periodic structure. We suggest a new boundary layer method. Our approach employs the homogenization method for periodic structures and takes into account additional boundary layer-type solutions. This approach can be used in the analysis of a tunnel crack in the fiber-reinforced composites or a plane crack in the laminated composites. The suggested method can be also of use in various contact problems in the field of elastic composites.

## INTRODUCTION

Let us consider the problem of a straight-line crack in the composite elastic medium with periodic structure in the case of rectangular periodicity cell (Fig. 1). We use the asymptotic integration method for the two-dimensional elasticity equations with rapidly varying periodic coefficients (see Kalamkarov et al (1)) and assume the two-scale expansions for the displacement vector into the power series of a small parameter $\varepsilon$, associated with the dimension of the periodicity cell (Fig.1). This method makes it possible not only to obtain the effective material parameters, but also to find to a high accuracy the stresses and displacements microstructures using the solution of auxiliary local problems on the periodicity cell and the solution of the boundary-value problem for the homogenized material.

* Department of Mathematics,
Moscow Institute of Chemical Engineering.

## A Crack in the Laminated Composite

The homogenization method for periodic structures (1) is applied, in particular, to the problem of the plane deformation of a laminated composite with a straight-line crack making an angle $\pi/2 - \varphi$ with the composite layers (Fig. 2). Assuming the uniform normal load $p$ at the lips of the crack, the stress intensity coefficients at the tip of the crack may be defind by

$$K_I = \lim_{x_1 \to a} \left[\sqrt{2\pi(x_1 - a)}\, \sigma_{22}(x_1, 0)\right] = p\sqrt{\pi a}\,\omega \quad (1)$$

For the case of shear stress $q$ at the lips we have

$$K_{II} = \lim_{x_1 \to a} \left[\sqrt{2\pi(x_1 - a)}\, \sigma_{12}(x_1, 0)\right] = q\sqrt{\pi a}\,\gamma \quad (2)$$

where $\omega$ and $\gamma$ depend on the location of the tip and on the elastic characteristics at the point $x_1 = a$ (Fig.2). The corresponding expressions are obtained by Parton and Kudryavtsev (2). We have, in particular,

$$\gamma = 1 \, , \quad \omega = \lim_{x_1 \to a} \omega(y_1) \quad (3)$$

$$\omega(y_1) = \frac{C'_{22}(y_1)\left[1 + S_1 S_2 \tilde{C}_{12} \tilde{C}_{11}^{-1}\right] - C'_{21}(y_1)\tilde{C}_{12}\tilde{C}_{11}^{-1}\left[1 + S_1 S_2 \tilde{C}_{22} C_{12}^{-1}\right]}{\tilde{C}_{22}\left[1 - \tilde{C}_{12}^{2}\tilde{C}_{11}^{-1}\tilde{C}_{22}^{-1}\right]}$$

for a crack perpendicular to the composite layers($\varphi$=0). Here elastic coefficients $C_{ij}(y_1)$ are 1-periodic functions of $y_1 = x_1/\varepsilon$ , $\tilde{C}_{ij}$ are the effective parameters of the homogenized transversally isotropic medium (1)

$$\tilde{C}_{11} = (\langle C_{11}^{-1} \rangle)^{-1}, \; \tilde{C}_{22} = \langle C_{11} - C_{12}^{2} C_{11}^{-1} \rangle + (\langle C_{12} C_{11}^{-1} \rangle)^2 (\langle C_{11}^{-1} \rangle)^{-1}$$
$$\tilde{C}_{12} = \langle C_{12} C_{11}^{-1} \rangle (\langle C_{11}^{-1} \rangle)^{-1}, \; \tilde{C}_{44} = \langle C_{44} \rangle, \; \tilde{C}_{55} = (\langle C_{44}^{-1} \rangle)^{-1} \quad (4)$$

Here symbol $\langle \cdots \rangle$ denotes integration over the layer,

$$C'_{22}(y_1) = C_{11} - \frac{C_{12}^2}{C_{11}} + \frac{C_{12} \cdot \tilde{C}_{12}}{C_{11}}, \quad C'_{21}(y_1) = \frac{C_{12} \cdot \tilde{C}_{11}}{C_{11}}$$

and $S_1$ and $S_2$ are the roots of the characteristic equation

$$\tilde{C}_{22} S^4 + \left[(\tilde{C}_{11}\tilde{C}_{22} - \tilde{C}_{12}^2)\tilde{C}_{55}^{-1} - 2\tilde{C}_{12}\right]S^2 + \tilde{C}_{11} = 0$$

A Boundary Layer Method in Fracture Mechanics of Composites. The formulae (3) are obtained in (2) in neglect of the boundary microeffect occuring in the neighbourhood of a crack, which is the reason why we propose more rigorous approach to the problem. Our approach employs the homogenization method for periodic structures (1) and takes into account additional boundary layer-type solutions.

To satisfy the mixed boundary conditions at the boundary $x_2 = 0$ of the periodically nonhomogeneous half-plane $x_2 > 0$ (Fig.1), two auxiliary problems are first considered. Let us consider the equation

$$\frac{\partial \sigma_{i\alpha}^{(\varepsilon)}(x_1, x_2)}{\partial x_\alpha} = 0, \quad \sigma_{i\alpha}^{(\varepsilon)} = C_{i\alpha k\beta}(y_1, y_2) \frac{\partial u_k^{(\varepsilon)}(x_1, x_2)}{\partial x_\beta} \qquad (5)$$

Here $\sigma_{i\alpha}^{(\varepsilon)}$ are the stresses; $u_k^{(\varepsilon)}$ the displacements; the elastic coefficients $C_{i\alpha k\beta}(y_1, y_2)$ are 1-periodic functions of $y_1 = x_1/\varepsilon$, $y_2 = x_2/\varepsilon$; Latin indices range from 1 to 3, Greek indices from 1 to 2, and summation convention for repeated indices is applied.

For the first of the above mentioned auxiliary problems we assume

$$\sigma_{i2}^{(\varepsilon)}(x_1, 0) = p_i(x_1), \quad (i = 1, 2, 3) \qquad (6)$$

at the boundary $x_2 = 0$. For the second problem, the mixed boundary conditions

$$\sigma_{i2}^{(\varepsilon)}(x_1, 0) = p_i(x_1), \quad (i = 1, 3), \quad u_2^{(\varepsilon)}(x_1, 0) = v_2(x_1) \qquad (7)$$

are assumed instead of (6).

For the problem, equation(5) and (6), the asymptotic representation for the solution is found to be:

$$u_k^{(\varepsilon)} = u_k^{(0)}(x_1, x_2) + \varepsilon \left[ N_{nk\beta}(y_1, y_2) + N_{nk\beta}^{(1)}(y_1, y_2) \right] \frac{\partial u_k^{(0)}}{\partial x_\beta} + O(\varepsilon^2) \qquad (8)$$

$$\sigma_{ij}^{(\varepsilon)} \approx \sigma_{ij}^{(0)} = \left[ C_{i\alpha k\beta} + C_{i\alpha n\gamma} \frac{\partial}{\partial y_\gamma} \left( N_{nk\beta} + N_{nk\beta}^{(1)} \right) \right] \frac{\partial u_k^{(0)}}{\partial x_\beta}$$

Here $u_k^{(0)}(x_1, x_2)$ is the solution of the homogenized problem

$$\tilde{C}_{i\alpha k\beta} \frac{\partial^2 u_k^{(0)}}{\partial x_\alpha \partial x_\beta} = 0, \quad \tilde{C}_{i\alpha k\beta} = \int_0^1 \int_0^1 \left( C_{i\alpha k\beta} + C_{i\alpha n\gamma} \frac{\partial N_{nk\beta}}{\partial y_\gamma} \right) dy_1 \, dy_2 \qquad (9)$$

$$\hat{C}_{i2k\beta} \frac{\partial u_k^{(0)}}{\partial x_\beta} \bigg|_{x_2 = 0} = p_i(x_1), \quad (i = 1, 2, 3) \qquad (10)$$

$$\hat{C}_{i2k\beta} = \int_0^1 C_{i2k\beta}^{*}(y_1) \, dy_1, \quad C_{i2k\beta}^{*} = \left( C_{i2k\beta} + C_{i2n\gamma} \frac{\partial N_{nk\beta}}{\partial y_\gamma} \right) \bigg|_{y_2 = 0}$$

Where $N_{nk\beta}(y_1, y_2)$ are 1-periodic functions of $y_1$, $y_2$ defined as solutions of the local problem

$$\frac{\partial}{\partial y_\alpha}\left(C_{i\alpha n\gamma}\frac{\partial N_{nk\beta}}{\partial y_\gamma}\right)_{(a)} = -\frac{\partial C_{i\alpha k\beta}}{\partial y_\alpha}, \quad \langle N_{nk\beta}\rangle = 0 \qquad (11)$$

The functions $N_{nk\beta}(y_1, y_2)$ are 1-periodic only in $y_1$ and are determined from the boundary-layer problem

$$\frac{\partial}{\partial y_\alpha}\left(C_{i\alpha n\gamma}\frac{\partial N^{(1)}_{nk\beta}}{\partial y_\gamma}\right) = 0$$

$$\left(C_{i2n\gamma}\frac{\partial N^{(1)}_{nk\beta}}{\partial y_\gamma}\right)\Bigg|_{y_2=0} = \hat{C}_{i2k\beta} - C^{*}_{i2k\beta}, \quad (i=1,2,3) \qquad (12)$$

$$N^{(1)}_{nk\beta} \to 0 \quad \text{for} \quad y_2 \to \infty$$

For the second problem, equation(5) and (7), we have obtained

$$u_k^{(\varepsilon)} = u_k^{(0)}(x_1, x_2) + \varepsilon\left[N_{nk\beta} + N^{(2)}_{nk\beta}(y_1, y_2)\right]\frac{\partial u_k^{(0)}}{\partial x_\beta} + O(\varepsilon^2) \qquad (13)$$

$$\sigma^{(\varepsilon)}_{i\alpha} \approx \sigma^{(0)}_{i\alpha} = \left[C_{i\alpha k\beta} + C_{i\alpha n\gamma}\frac{\partial}{\partial y_\gamma}\left(N_{nk\beta} + N^{(2)}_{nk\beta}\right)\right]\frac{\partial u_k^{(0)}}{\partial x_\beta}$$

Here $u_k^{(0)}(x_1, x_2)$ is the solution of the homogenized problem (9) under the boundary conditions

$$\hat{C}_{i2k\beta}\frac{\partial u_k^{(0)}}{\partial x_\beta}\Bigg|_{x_2=0} = p_i(x_1), \ (i=1,3), \ u_2^{(0)}(x_1, 0) = v_2(x_1) \qquad (14)$$

The functions $N^{(2)}_{nk\beta}(y_1, y_2)$ are 1-periodic only in $y_1$ and are determined from the boundary-layer problem

$$\frac{\partial}{\partial y_\alpha}\left(C_{i\alpha n\gamma}\frac{\partial N^{(2)}_{nk\beta}}{\partial y_\gamma}\right) = 0$$

$$\left(C_{i2n\gamma}\frac{\partial N^{(2)}_{nk\beta}}{\partial y_\gamma}\right)\Bigg|_{y_2=0} = \hat{C}_{i2k\beta} - C^{*}_{i2k\beta}, \quad (i=1,3) \qquad (15)$$

$$N^{(2)}_{2k\beta}(y_1, 0) = -N_{2k\beta}(y_1, 0)$$

$$N^{(2)}_{nk\beta} \to 0 \quad \text{for} \quad y_2 \to \infty$$

It has been proved that the problems (11),(12), (15) have unique solutions.

For the determination of the microstresses in the neighbourhood of a crack, the auxiliary domain IV, containing the end of a crack, is considered (Fig. 3).

The problem we address next is to match the domain IV solution with the above two types of the boundary-layer solutions at the vertical boundaries, namely, with the solution of the problem (8)-(12) in the domain I and solution of the problem (13)-(15) in the domain II (see Fig.3).

The functions $N_{nk\beta}^{(1)}$ and $N_{nk\beta}^{(2)}$ are the boundary-layer solutions and can therefor be neglected in the domain III (Fig.3).

For the domain IV containing only a few periodicity cells, the solution of the problem (5) can be found numerically.

By the application of the above approach to the plane crack problem in a laminated composite, more accurate values for stress intensity coefficients (2), (3) have been obtained.

In conclusion, it has proved possible by considering the domain IV with known boundary conditions to compute with sufficient accuracy the stress fields arising in the neighbourhood of a crack with an arbitrary located tip and, in particular, for the tip at the interface between the components of the composite material.

## REFERENCES

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(2) Parton, V.Z. and Kudryavtsev, B.A., Phys.-Chem. Mech. of Materials, No. 1, 1986, pp. 76-84.

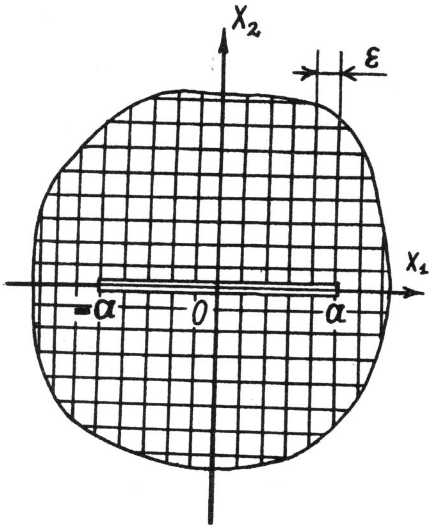


Figure 1 A crack in the composite material

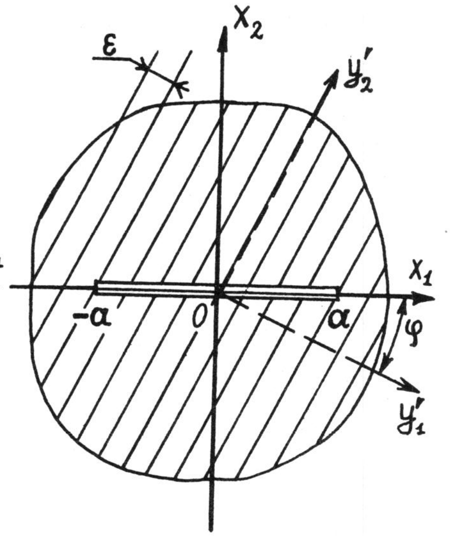


Figure 2 A crack in the laminated composite

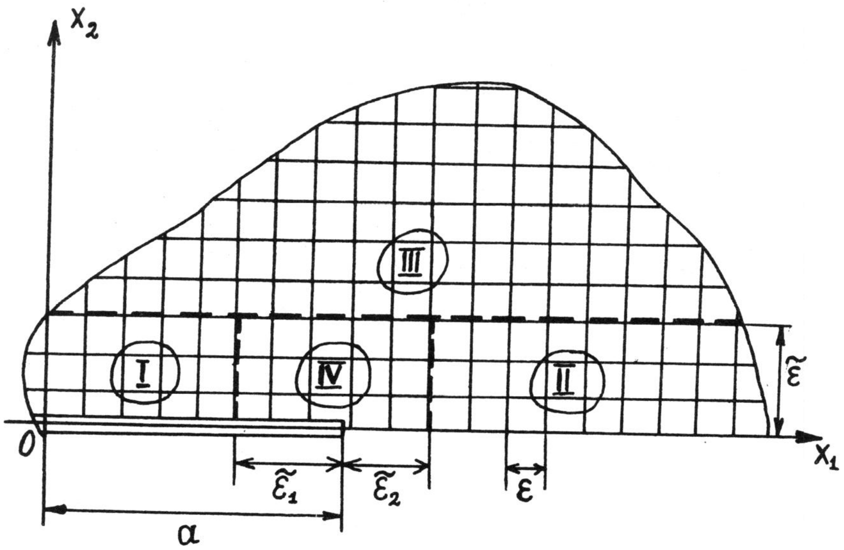


Figure 3 The auxiliary domains I-IV in the neighbourhood of a crack in the composite material