A BOUNDARY INTEGRAL EQUATION METHOD FOR THE PROBLEM OF MULTIPLE, INTERACTING CRACKS IN ANISOTROPIC MATERIALS

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A boundary element method is described which combines the features of an advanced BEM-code with special analytical techniques for solving two-dimensional crack problems in elastically anisotropic media. Cracks are modelled by special Green's functions, or alternatively, by special singular crack tip elements. On the basis of a subdomain technique, the Green's function for two collinear semi-infinite cracks is used for modelling finite double cracks. In general, elastic anisotropy causes the coupling of all three stress intensity factors. Its influence is demonstrated by an example including eight cracks.

INTRODUCTION

Elastic interaction of cracks located in a domain which is composed of subdomains with different elastic properties including anisotropy is of particular interest in the mechanics of materials. Different parts of a structure often consist of different materials, or the material is the same but the orientation is different (grain structure of metals). A crack may lie entirely inside a homogeneous region, it may be situated along the interface between two materials, or it may cross such an interface. Mathematical investigations of multiple crack problems are not easy, and there are only few explicit solutions to special configurations even in the two-dimensional case. Violaton et al. (1) derived the Green's function for two collinear cracks of equal length in the elastically isotropic unbounded plane and then investigated the interaction of the cracks under

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different loading conditions. Tsai (2) found a method of calculating stress intensity factors of two collinear cracks situated symmetrically in an infinite strip of orthotropic material. For more complex configurations a computorized numerical procedure must be used, the most popular one being the finite element method (FEM). As an alternative, during the last decade boundary element methods (BEM) have become important in continuum mechanics. There are several types of boundary element methods which are all based upon an integral equation formulation of the boundary value problem. Snyder and Cruse (3) derived the Green's function for a finite straight crack in the unbounded plane and considered elastic anisotropy with the restriction that this plane must be a crystallographic plane of symmetry. Based on that function the authors implemented a simple variant of the direct BEM which they developed subsequently (4). Violaton et al. also used their Green's function for the double crack in a BEM-code to calculate stress intensity factors for a bounded plate containing two collinear cracks (5). Their method was a simple implementation of indirect BEM. Meanwhile Lachat and Watson (6) had developed some techniques which made direct BEM a powerful competitor of FEM in the field of elasticity problems of arbitrarily shaped bodies. Lachat and Watson introduced multiparametric elements, efficient numerical algorithms for element integration, and substructuring into BEM. Kuhn (7) was the first to use substructuring for solving multiple crack problems by direct BEM on the basis of the Green's function for one straight crack, followed by Rudolphi and Koo (8) and by Ang (9). If a Green's function for a finite crack is used, as it was done by the above authors, the interaction of two crack tips is the more falsified the closer they are together. This is due to the inevitable subdomain boundary between them. In the present paper a novel method is introduced which makes use of the Green's function for two collinear semi-infinite cracks in the unbounded plane. This function is much easier to handle than that by Violaton et al. Moreover, the present formulation also holds true for the most general elastic anisotropy, where there is no symmetry and no decoupling between plane strain or stress and antiplane strain. An effective two-dimensional direct BEM-code has been written which is now part of the BEM-library BOREAS (Bruchmechanisch Orientiertes Randelemente-Analyse-System, i.e. Boundary Element Analysis System with Fracture-mechanical Orientation) developed at the Institute of Solid State Physics and Electron Microscopy in Halle.

OUTLINES OF THE PROCEDURE

In the case of linear elastostatics, Betti's reciprocal theorem lies at the bottom of the formulation. If we use a fundamental solution (Green's function) as reference group, we are led to Somigliana's identity. This is a singular integral equation which determines the unknown boundary data provided that the boundary value problem is properly posed. In the BEM, the boundary is discretized by parametric elements with a number of boundary nodes and the integrals over the elements are evaluated separately (10). Substructuring leads to additional inner boundaries which are discretized in the same way. The present formulation generally uses isoparametric elements with quadratic shape functions modified only for special singular crack tip elements. The elements form open or closed subcontours within which they are conforming. Open subcontours are used generally for substructuring, and for modelling discontinuities of tractions or displacements. If two or more open subcontours end in the same mathematical point, the collocation points corresponding with the nodes in identical position must be shifted into the interior of the elements in order to avoid mathematical difficulties. As a result of the outlined procedure, the integral equation is transformed into a set of linear algebraic equations, which connect the unknown boundary node values.

INCORPORATION OF CRACKS

Traction-free straight cracks which are not situated at an inner boundary between two subregions are easily incorporated by special Green's functions. In this case the crack faces need not be discretized at all. Fig. 1 illustrates the crack configurations for which Green's functions are available in a relatively simple mathematical form (11). Fig. 2 shows the discretization for a problem of 8 multiple, interacting cracks which are modelled by Green's functions of type 2. For solving edge crack problems the fundamental solution for a semi-infinite crack is advantageous. The Green's function for two collinear semi-infinite cracks is used if two crack tips are lying close together. Fig. 3 shows how a double crack is incorporated into a domain of arbitrary geometry. In the inner subdomain the Green's function for an infinite double crack is employed while the boundary integral formulation in the surrounding one is based on the Green's function for a finite crack. Thus, the result is exact irrespective of the distance of the cracks. The stress intensity factors KI, KII, KIII are calculated by evaluating regular integrals after solving the boundary value problem. If crack faces are partially loaded, their respective parts must be discretized. The method of using special Green's functions fails, if crack faces are loaded at the very crack tip. Such cases are properly treated by applying special singular crack tip elements. They are similar to the quarter-point elements used in FEM (10), but they explicitly include the $1/\sqrt{r}$-singularity of the crack tip stresses. Here, the stress intensity factors KI, KII, KIII are directly gained from the traction-values of the crack-tip nodes. The same method applies when a crack is located at the interface between two different materials. For the isotropic case Hein and Erdogan (13) showed that the order of the singularity is the same as in homogeneous materials. So the same boundary elements can be used.

ELASTIC ANISOTROPY

If the elastic plane is not a crystallographic plane of symmetry there is no decoupling between plane strain or stress and antiplane strain. This happens if the material itself has no symmetry (triclinic material), or if it is more symmetrical but has no symmetrical orientation. In these cases all the stress intensity factors KI, KII, KIII must be considered simultaneously. Based on a recent, very general stress function representation (14), Green's functions for the crack configurations of fig.1 were derived, which also apply to the most general elastic anisotropy (11).

PERIODICITY

Periodicity is easily formulated by identifying the displacement and traction values of corresponding nodes. For instance, the lower and the upper boundary of the example of fig. 2 might be "periodicity lines". Periodicity often implies a rigid translation and rotation of a periodicity line. Both the latter are determined by the total forces and the bending moment, which act on that line. This option was used, for instance, for solving the problem of thermo-shock induced growth of periodically arranged parallel cracks in an infinite strip of a brittle material (15).

NUMERICAL EXAMPLES

Violaton et al. (5) considered an isotropic quadratic sheet with two symmetrically arranged collinear cracks under constant tension t. The ratio of the crack length L to the width of the sheet 2W was 0.2. The authors calculated the stress intensity factors KI of inner (i) and outer (o) crack tips for various distances of inner crack tips 2k. Table 1 shows the normalized results HI=KI/(t$\sqrt{\pi L}$) given in (5) compared with own ones calculated on the basis of a discretization with 50 nodes of the type outlined in fig. 3. A second example demonstrates the influence of anisotropy. Consider a quadratic sheet of an anisotropic material containing 8 cracks, which are arranged with 4 axes of symmetry according to fig.2. Again the width of the sample is 2L. Let the upper right crack be crack number one. Its tips have the following positions: xi=0.2L, yi=0.1L, xo=0.7L, yo=0.35L. The sample is of potassium bromide as a model substance which is relatively strongly anisotropic with C11=34470MPa, C12=4790MPa, C44=5080MPa. The material is subsequently turned out of its symmetrical position by three rotations around the x-, y-, and z-axes at angels of 30, 20, and 10 degrees, respectively. The sample is stretched with its clamped ends uniformly displaced in x-direction (to the right in fig. 2). Table 2 shows the normalized stress intensity factors HI, HII, HIII of the cracks in the upper half of the sample. Here, t is the averaged total traction at the clamped ends. Under this loading, the results in the lower half are the same with HIII, however, having opposite signs.

TABLE 1 - Normalized stress intensity factors for double cracks of equal length in a quadratic sheet under constant tension.

k/W	Violaton et al (5) HI(i)	Violaton et al (5) HI(o)	Author HI(i)	Author HI(o)
0.0	-	-	-	1.055
0.001	-	-	2.208	0.885
0.01	1.089	0.818	1.089	0.818
0.05	0.809	0.763	0.809	0.763
0.1	0.759	0.744	0.759	0.744
0.2	0.735	0.730	0.732	0.729
0.3	0.726	0.724	0.723	0.722
0.4	0.722	0.722	0.720	0.720

TABLE 2 - Normalized stress intensity factors for eight cracks in an anisotropic quadratic sheet under tension with its ends clamped.

	crack 1	crack 2	crack 3	crack 4
HI (i)	0.205	0.484	0.503	0.164
HII (i)	-0.489	-0.339	0.384	0.394
HIII(i)	0.163	0.077	0.071	-0.003
HI (o)	0.002	0.837	0.713	0.020
HII (o)	-0.188	-0.307	0.315	0.179
HIII(o)	0.100	0.159	0.062	-0.022

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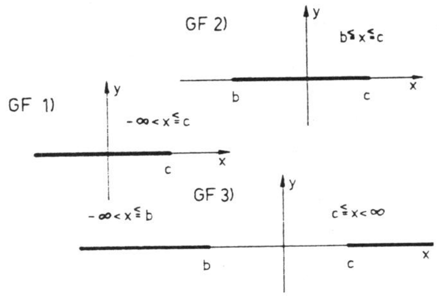

Figure 1. Green's functions for straight cracks

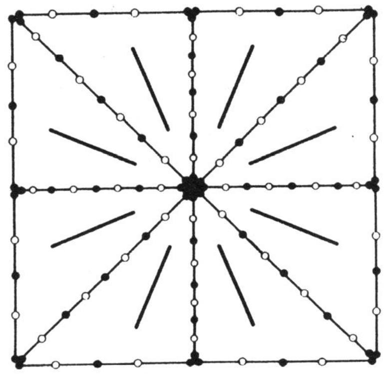

Figure 2. Discretization of a sample with eight cracks

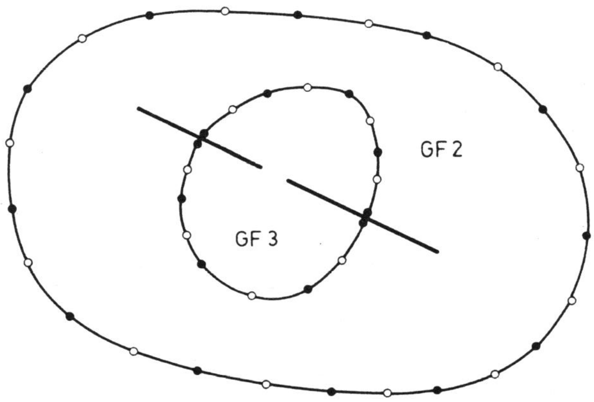

Figure 3. Discretization of a region with a double crack