NUMERICAL TREATMENT OF FRACTURE MECHANICS PROBLEMS

V.Z. Parton* and V.G. Boriskovsky**

Some results of recent investigations on finite element method (FEM) application for dynamic fracture mechanics (DFM) problems and a review of the main trends in fracture mechanics numerical methods in the USSR are given. A procedure for higher terms calculation in stress and displacement fields representations near the crack tip is proposed. These terms have a strong influence upon the crack trajectory and crack branching as shown by the numerous data.

INTRODUCTION

The actuality of fracture mechanics numerical methods does not need special judgement. They have been developed in the USSR since the beginning of the 70th years and embrace a number of important trends described in the authors' monographies and other publications (1-5). The solution of static and dynamic problems of linear and elastic-plastic fracture mechanics in two and three dimensions with the use of FEM, boundary integral equations method (BIEM) and weighting function methods (WFM) can be pinpointed.

The use of FEM is based on the introduction of singular finite elements (SFE) with special displacement fields approximations, proceeding from the analytical representations at the crack tip (2, 4, 6-10), and quarter-type degenerated isoparametric elements, which model stress singularity as well (3). While employing the BIEM, an approach based on the limit analyses of solutions for comparatively thin cavities was worked out (1, 5, 11). The combined numerical-analytical construction of the

* Department of Mathematics, Moscow Institute of Chemical Engineering
** Department of Automatized Design, Donetsk Research and Design Institute of Civil Engineering.

weight functions enables to advance the procedure (12, 13) of stress intensity factor (SIF) determination in the three-dimensional case. The method of SIF computing under harmonic loading is founded on the representation of these factors by superposition of the model SIF corresponding to the normalised free vibration modes, which are determined from the FEM solution (2, 4, 14-17).

It was recently noted in some papers (see, for example ref. (18)) that the use of the asymptotic stress and displacement representations near the crack tip accounting only for the leading terms is insufficient and can cause contradiction. The second-order terms corresponding to the linear displacement alteration and the constant part of have a strong influence upon crack kinking and branching processes, as shown in ref. (19). A sufficient number of higher-order terms has to be accounted for in the photo elasticity data evaluation. The FEM with the use of SFE allows to calculate the terms of the power, which is less or equal than half of the element nodes, while the degenerated isoparametric element application gives the possibility to compute the terms of the first and second order.

The presented method of higher-order terms computing corresponds to the propagation crack case. The limit transition allows to apply it to the stationary crack case.

THE FEM APPLICATION TO THE DFM PROBLEMS

At present the variety of the FEM approaches to the numerical solution of linear fracture mechanics problems is established. While employing one of them we introduce a special finite element at the crack tip with displacement fields approximations taken from the analytical solution for the cracked region.

This approach has been described in scientific literature in detail (2, 5). We should like to emphasize here that Soviet authors' elaboration in this direction was among the first (6-10). The SFE for plane static problems was suggested in (6), while a similar one was introduced in (7, 8) for the Kirchhoff-Love plates. The SFE idea was transferred to the problems of plates lying on an elastic foundation in (9), and to the problem of cracked shells - in (10) -. The convenient method of stress intensity factors determination in the case of cracked plates under harmonic loading was proposed in (14-17).

One has the following equation for the stress intensity factor:

$$K(t) = K_s \sum_i Z_i \omega_i^2 (\omega_i^2 - \omega^2)^{-1} e^{i\omega t} \quad \text{..........(1)}$$

$$Z_i = K^{(i)} [\{x_i^{(i)}\}^T \{f\}] / K_s \omega^2 \quad \text{..........(2)}$$

$$\sum Z_i = 1 \qquad \text{(3)}$$

Unfortunately, the analytical solutions known in literature correspond to unbounded bodies and give no information for comparison with numerical results obtained according to (1).

The numerical solution error estimation follows from (3). This equation really states that $K_s = K$ when $\omega = 0$.

Consider one of the results obtained for the square plate with central crack (the crack length and plate side ration was 0.364). The static calculations lead to $K_s / \sigma \sqrt{\pi \ell} = 1.23$.
When solving the dynamic problem 16 modes were taken into account.

In the case of homogeneous tension-compression only modes 5, 8, 9, 13 contribute to the sum (1), and the corresponding expression is:

$$K(t) = K_s \left[\frac{0.920}{1 - \left(\frac{\omega}{4.98}\right)^2} - \frac{0.102}{\left(1 - \frac{\omega}{9.331}\right)^2} \right.$$
$$\left. \frac{0.212}{1 - \left(\frac{\omega}{11.089}\right)^2} - \frac{0.014}{1 - \left(\frac{\omega}{27.394}\right)^2} \right] e^{i\omega t} \qquad \text{(4)}$$

One can see that the error is 1.6% when $\omega = 0$, and that the SIF amplitude monotonically increases when ω rises. The amplitude becomes unbounded, if the frequency approaches to ω_1.

THE SECOND-ORDER COMPUTATION

The general solution for the stress and displacement fields near the tip of a running mode I-II crack (Fig. 1) can be represented by the eigen functions superposition (4, 20):

$$\sigma_{xn} = \frac{K_n^S B_I(c)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ (1 + 2\beta_1^2 - \beta_2^2) r_1^{\frac{n}{2}-1} \cos\left(\frac{n}{2}-1\right)\Theta_1 + \right.$$
$$\left. - 2h(n) r_2^{\frac{n}{2}-1} \cos\left(\frac{n}{2}-1\right)\Theta_2 \right\} +$$
$$+ \frac{K_n^A B_{II}(c)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ (1 + 2\beta_1^2 - \beta_2^2) r_1^{\frac{n}{2}-1} \sin\left(\frac{n}{2}-1\right)\Theta_1 + \right.$$
$$\left. - 2\bar{h}(n) r_2^{\frac{n}{2}-1} \sin\left(\frac{n}{2}-1\right)\Theta_2 \right\} \qquad \text{(5)}$$

$$\sigma_{yn} = \frac{K_n^S B_I(c)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ -(1+\beta_2^2) r_1^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\Theta_1 + 2h(n) r_2^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\Theta_2 \right\} + \frac{K_n^A B_{II}(c)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ -(1+\beta_2^2) r_1^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\Theta_1 + 2h(\bar{n}) r_2^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\Theta_2 \right\} \quad \text{........(6)}$$

$$\sigma_{xyn} = \frac{K_n^S B_I(c)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ -2\beta_1 r_1^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\Theta_1 + \frac{1+\beta_2^2}{\beta_2} h(n) r_2^{\frac{n}{2}-1} \sin(\tfrac{n}{2}-1)\Theta_2 \right\} + \frac{K_n^A B_{II}(c)}{\sqrt{2\pi}} \frac{n(n+1)}{2} \left\{ 2\beta_1 r_1^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\Theta_1 - \frac{1+\beta_2^2}{\beta_2} h(\bar{n}) r_2^{\frac{n}{2}-1} \cos(\tfrac{n}{2}-1)\Theta_2 \right\} \quad \text{........(7)}$$

$$u_n = \frac{K_n^S B_I(c)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ r_1^{\frac{n}{2}} \cos\tfrac{n}{2}\Theta_1 - h(n) r_2^{\frac{n}{2}} \cos\tfrac{n}{2}\Theta_2 \right\} + \frac{K_n^A B_{II}(c)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ r_1^{\frac{n}{2}} \sin\tfrac{n}{2}\Theta_1 - h(\bar{n}) r_2^{\frac{n}{2}} \sin\tfrac{n}{2}\Theta_2 \right\} \quad \text{....................(8)}$$

$$v_n = \frac{K_n^S B_I(c)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ -r_1^{\frac{n}{2}} \beta_1 \sin\tfrac{n}{2}\Theta_1 + \frac{h(n)}{\beta_2} r_2^{\frac{n}{2}} \sin\tfrac{n}{2}\Theta_2 \right\} + \frac{K_n^S B_{II}(c)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ \beta_1 r_1^{\frac{n}{2}} \cos\tfrac{n}{2}\Theta_1 - \frac{h(\bar{n})}{\beta_2} r_2^{\frac{n}{2}} \cos\tfrac{n}{2}\Theta_2 \right\} \quad \text{....................(9)}$$

$$B_I(c) = \frac{1+\beta_2^2}{D} \qquad \text{(10)}$$

$$B_{II}(c) = \frac{2\beta_2}{D} \qquad \text{(11)}$$

$$D = 4\beta_1\beta_2 - (1+\beta_2^2)^2 \qquad \text{(12)}$$

$$r_j e^{i\theta_j} = x + i\beta_j y \qquad \text{(13)}$$

$$\beta_j^2 = 1-(c/c_j)^2 \qquad \text{(14)}$$

$$h(n) = \begin{cases} 2\beta_1\beta_2/(1+\beta_2^2), & n=2k+1 \\ (1+\beta_2^2)/2, & n=2k \end{cases} \qquad \text{(15)}$$

$$\bar{n} = n+1 \qquad \text{(16)}$$

The terms corresponding to n=2 are :

$$\sigma_{x2} = \frac{6K_2^S B_I(c)(\beta_1^2-\beta_2^2)}{\sqrt{2\pi}} \qquad \text{(17)}$$

$$\sigma_{y2} = \sigma_{xy2} = 0 \qquad \text{(18)}$$

$$u_2 = \frac{3K_2^S B_I(c)}{2\mu}\sqrt{\frac{2}{\pi}}\left\{r_1\cos\theta_1 - \frac{1+\beta_2^2}{2}r_2\cos\theta_2\right\} + \frac{3K_2^A B_{II}(c)}{2\mu}\sqrt{\frac{2}{\pi}}\left\{r_1\sin\theta_1 - \frac{2\beta_1\beta_2}{1+\beta_2^2}r_2\sin\theta_2\right\} \qquad \text{(19)}$$

$$v_2 = \frac{3K_2^S B_I(c)}{2\mu}\sqrt{\frac{2}{\pi}}\left\{-\beta_1 r_1\sin\theta_1 + \frac{1+\beta_2^2}{2\beta_2}r_2\sin\theta_2\right\} + \frac{3K_2^A B_{II}(c)}{2\mu}\sqrt{\frac{2}{\pi}}\left\{\beta_1 r_1\cos\theta_1 - \frac{2\beta_1}{1+\beta_2^2}r_2\cos\theta_2\right\} \qquad \text{(20)}$$

Now it can be shown how the constant part of the stress can be computed using the FEM results. The approximation along the 1-2 side has the following form when the quartertype isoparametric quadratic degenerate element is used (Fig. 2) (3, 4).

$$u = u_{(1)} + \sqrt{\frac{x}{e_1}}\left(-3u_{(1)} - u_{(2)} + 4u_{(5)}\right) + \frac{x}{e_1}\left(2u_{(1)} + 2u_{(2)} - 4u_{(5)}\right) \qquad \text{..........(21)}$$

Comparing the r-coefficients in (21) and (19) when $\theta = 0$ one has:

$$\frac{3K_2^S B_I(c)}{2\mu}\sqrt{\frac{2}{\pi}}\,\frac{1-\beta_2^2}{2} = \frac{2u_{(1)} + 2u_{(2)} - 4u_{(5)}}{e_1} \qquad \text{.....(22)}$$

Denoting $\Delta_1 = 2u_{(1)} + 2u_{(2)} - 4u_{(5)}$ and using (22) we obtain from (17) that:

$$\sigma_{x2} = \frac{4\Delta_1\mu}{e_1}\,\frac{\beta_1^2-\beta_2^2}{1-\beta_2^2} = \frac{4\Delta_1\mu}{e_1}\left(1-\frac{c^2}{C_1^2}\right) \qquad \text{.....(23)}$$

$$\sigma_{x2} = \frac{4\Delta_1}{e_1}\,\mu\,\frac{\lambda+\mu}{\lambda+2\mu} \qquad \text{...............(24)}$$

It is obvious that this relation preserves its form in the stationary crack case.

One can obtain a similar relation comparing the v-approximation along the 1-4 side and (19) when $\theta = \pi/2$

$$\sigma_{x2} = \frac{4\Delta_2}{e_2}\,\mu\,\frac{\lambda+\mu}{\lambda} \qquad \text{...............(25)}$$

Another procedure can be used in the SFE case. If the displacement approximation has the form of the eigen functions (19), (20) superposition then the vector of the element nodes displacement is equal to:

$$\{\delta\} = [L]\{K_0^S, K_0^A, K_1^S, K_1^A, K_2^S, \ldots\}^T \qquad \text{......(26)}$$

The matrix $[L]$ is obtained by the nodes coordinates substitution into (8), (9). Denoting $\{R\}$ - the 5th line of $[L]^{-1}$ one can write:

$$K_2^S = \{R\}\{\delta\} \qquad \text{...............(27)}$$

and after substitution into (17):

$$\sigma_{x2} = \frac{6K_2^S}{\sqrt{2\pi}}\,\frac{(1+\beta_2^2)(\beta_1^2-\beta_2^2)}{4\beta_1\beta_2-(1+\beta_2^2)^2} \qquad \text{...............(28)}$$

In the limit case the relation for the stationary crack can be deduced:

$$\sigma_{x_2} = \frac{\sigma K_2^{\,s}}{\sqrt{2\pi}} \qquad (29)$$

THE WFM APPLICATION

Let us dwell on the key issue of using the data obtained from some basic problem solutions for the range of problems with other loading condition. Having deduced the WF the constructive approach can be obtained. The convenient way for weight function determination in the plane case was shown in (21), however its application to the space case is accompanied with significant difficulties. Particularly we have to know the crack profile, but in the majority of scientific works (see review (22)) the profile equation does not satisfy both the energy balance equation and asymptotic solution. An interesting attempt of WF determination for space problems was made in (23), but the authors obtained qualitatively incorrect results.

The method of WF determination based on the combination of basic problems, numerical solutions and fundamental equations was proposed in (12, 13). The SIF distribution along a crack front $\bar{K}_n$ under nonhomogeneous loading $\bar{\sigma}_n$ satisfies the following integral equation:

$$\frac{1}{H}\int_{\Gamma} \bar{K}_0 \bar{K}_n \delta S \, d\Gamma = \int_{S} \bar{\sigma}_n \delta \bar{u}_0 \, dS \qquad (30)$$

The explicit dependence between u_0 and $f(\Theta, a/b)$ - SIF transformed to the dimensionless form, satisfying the asymptotic equation, energy balance equation and inseparability condition was obtained for elliptical cracks (a, b - the ellipse axes).

The SIF for cracks of different types under nonhomogeneous loading with account of finite geometries, mutual crack influence etc. can be obtained after determination of f_0 using the numerical solution.

The problem of a subsurface semi elliptic crack in the half-space was considered in detail. The f_0 - data were determined on the results of (24), where the solution was obtained by means of singular integral equations method.

Three cases of nominal loading in the form

$$\bar{\sigma} = \bar{\sigma}_0 \left(\frac{x}{a}\right)^n \qquad \text{.....................(31)}$$

were investigated. The values of the SIF for two characteristic points:

$$C_A = \frac{f_n(\pi/2, a/b)}{f_0(\pi/2, a/b)},\ C_B = \frac{f_n(0, a/b)}{f_0(0, a/b)};\ (1 \leq n \leq 3) \qquad \text{...(32)}$$

are listed in (12, 13).($\Theta = \pi/2$ is the point of maximal depth).

There are few data for distribution along the subsurface crack front when $n > 1$, so FEM calculations were made for comparison. The FEM results were obtained for the semicircular crack depth a/H = 0.2. The grid containing 440 prismatic solid elements was used. The error of SIF numerical determination was 3% in the case of homogeneous loading (when compared with data taken from (24), and it increased to 10% in the nonhomogeneous loading case.

The values of C_A, C_B obtained with the help of (30), are in good correspondence with FEM results, and the accuracy of the numerical solution is not less than in the FEM case.

THE BIEM APPLICATION

The general approach based on the potential method can not be directly applied to cracked bodies, because the associated problems become degenerate. Hence an approach was introduced, connected with the replacement of the crack by an elliptical cavity of finite width (1, 5, 11). The limit analyses of solutions for such cavities give the crack solution. The crack tip stresses are of the following form:

$$\sigma_{ij} = \alpha_1 \left(\frac{r}{\ell}\right)^{-1/2} f_{1ij}(\Theta) + \alpha_2 \left(\frac{r}{\ell}\right)^{0} f_{2ij}(\Theta) + \alpha_3 \left(\frac{r}{\ell}\right)^{1/2} f_{3ij}(\Theta) + \cdots \qquad \text{.....................(33)}$$

where $\alpha_1 = K/\sqrt{2\pi}$. One needs to determine stresses very closely to the crack tip, while using the first term in (33). But the SIF can be determined without crack tip stresses determination, if the next terms in (33) are taken into account. This statement can be illustrated by considering a plane with a thin elliptical cut under tension. The stresses along the longest axis are:

$$\sigma_y = \frac{(\ell + r)\sigma}{\sqrt{r^2 + 2r\ell}} \qquad \text{.....................(34)}$$

After multiplying with $\sqrt{r}$ and taking the r-series representation one has:

$$\sigma_y \sqrt{r} \simeq d_1 + d_2 r \quad \text{......................(35)}$$

$$d_1 = \lim_{r \to 0} \sigma \sqrt{r} \quad \text{......................(36)}$$

It was shown that the $\sigma_y \sqrt{r} \to r$ dependence can be approximated as a line, if r is comparatively large. The extrapolation to the point r = 0 leads to a K-value with 1.5% error, when r/ℓ = 0.02, and 7% when r/ℓ = 0.1. It is important to underline that BIEM application requires less computer capacities then finite element method.

SYMBOLS USED

λ, μ	= Lamé coefficients
H	= generalized Young's modulus
$\sigma_x, \sigma_y, \sigma_{xy}$	= stress tensor components
u, v	= displacement vector components
c_1, c_2	= elastic waves velocities
c	= crack velocity
$\{X^{(i)}\}$	= free vibration frequencies
ω_i	= free vibration modes
ω	= force frequency
$\{f\}$	= load vector
$K^{(i)}$	= SIF corresponding to
$K = K^s$	= static SIF
K_n^S, K_n^A	= coefficients of eigen functions
$u_{(i)}, v_{(i)}$	= element nodes displacements
Γ	= crack contour
S	= crack area
$\bar{K}_0$	= SIF under homogeneous load $\bar{\sigma}_0$ action
$\bar{u}_0$	= crack opening displacement under $\bar{\sigma}_0$ action

REFERENCES

(1) Parton, V.Z. and Perlin, P.I. Integral equations of the theory of elasticity. Moscow, Nauka, 1977 (in Russian).

(2) Boriskovsky, V.G. and Parton, V.Z. Dynamic fracture mechanics. Advances in Science and Technology of VINITI (Itogi Nauka i Techniki VINITI). Ser. Mechanics of Deformable Solids, vol. 16. Moscow, VINITI, 1983 (in Russian).

(3) Morosov, E.M. and Nikishkov, G.P. The finite element method in fracture mechanics. Moscow, Nauka, 1980 (in Russian).

(4) Parton, V.Z. and Boriskovsky, V.G. Dynamic fracture mechanics. Moscow, Mashinostroyeniye, 1985 (in Russian).

(5) Parton, V.Z. and Morosov, E.M. Elastic-plastic fracture mechanics. Moscow, Nauka, 1985 (in Russian).

(6) Dashevsky, E.M. In: Numerical Methods, algorithms and programs. Proc. TaNIISK, vol. 20, 1971, pp. 77-88 (in Russian).

(7) Dashevsky, E.M. and Boriskovsky, V.G. Strength of Mater., vol. 10, 1976, pp. 42-44 (in Russian).

(8) Dashevsky, E.M., Kopylov, V.M. and Boriskovsky, V.G. In: 7th Sci. Conf. Comput. Appl. Mech. Deform. Solids, Pt II, Tashkent, 1975, pp. 83-84 (in Russian).

(9) Boriskovsky, V.G. and Dashevsky, E.M. Appl. Mech., vol. 13, No. 9, 1977, pp. 124-127 (in Russian).

(10) Dashevsky, E.M. and Boriskovsky, V.G. In: Numerical methods for the structural mechanics problems solution. Kiev, KISI, 1978, pp. 72-76 (in Russian).

(11) Karpov, G.N., Kurnosov, N.V. and Parton, V.Z. Strength of Mater., vol. 7, 1982, pp. 3-5.

(12) Vainshtock, V.A. Strength of Mater., vol. 6, 1986 (in Russian) (in press).

(13) Vainshtock, V.A. and Varfolomeev. I.V. Strength of Mater., vol. 8, 1986 (in Russian) (in press).

(14) Boriskovsky, G.V. Appl. Math. and Mech. (PMM), vol.43, 1979, pp. 764-768 (in Russian).

(15) Boriskovsky, V.G. and Parton, V.Z. In: Improvement of chemical machinery constructions. Moscow, MIChM, 1982, pp. 20-22 (in Russian).

(16) Boriskovsky, V.G., Dashevsky, E.M. Denisov, L.D. and Parton, V.Z. Eng. Fract. Mech., vol. 16, No. 4, 1982, pp. 459-465 (in Russian).

(17) Parton, V.Z. and Boriskovsky, V.G. In: Adv. Fract. Res. Proc. 6th Int. Conf. (ICF6). New Delhi, 4-10 Dec. 1984. Vol. 2. Oxford e.a., 1984, pp. 1185-1190.

(18) Eftis, T. and Subramonian, N. Eng. Fract. Mech., vol. 10, 1978, pp. 43-67.

(19) Ramulu, M. and Kobayashi, A.S. Exp. Mech., vol. 23, No. 1, 1983, pp. 1-9.

(20) Nishioka, R., Atluri, S.N. Eng. Fract. Mech., vol. 18, No. 1, 1983, pp. 1-22.

(21) Rice, T.P. Int. J. Solids and Struct., vol. 8, No. 6, 1972, pp. 751-758.

(22) Nedelchev, N.I. Strength of Mater., vol. 12, 1985, pp. 46-54 (in Russian).

(23) Mattachek, C., Morawietz, P. and Munz, D. Int. J. Fract., vol. 23, No. 3 1983, pp. 201-212.

(24) Hayashi, K. and Abe, H. Int. J. Fract. vol. 16, No. 3, 1980, pp. 275-285.

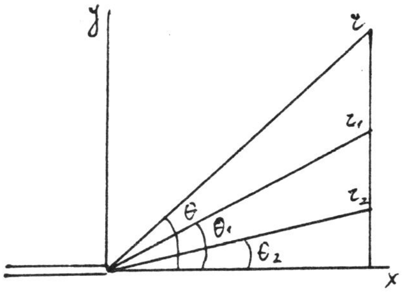

Fig. 1 The coordinates near the crack tip

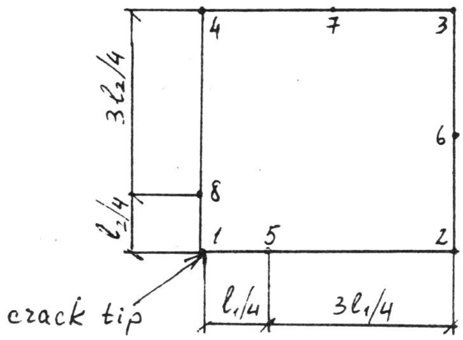

Fig. 2 The generated isoparametric element

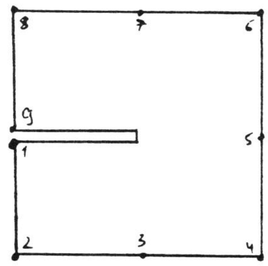

Fig. 3 The singular finite element