CONTINUUM MECHANICAL FOUNDATION OF CURVED THERMAL CRACK GROWTH IN ELASTO-PLASTIC MATERIALS

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A thermodynamically based theory has been derived to describe the thermal crack problem in elasto-plastic materials. The theory is founded upon Gibb's fundamental equation and the general energy balance equation. The coupling phenomenon between thermal and mechanical properties is discussed and a special form of the J-integral which is valid for elasto-plastic materials is presented. Moreover, a new approach to the internal energy has been introduced.

INTRODUCTION

The fracture process in solids is a dynamical process in a special sense and it depends on many microscopic parameters. The aim of this work is to formulate an energy criterion which is capable of describing the onset, the path and the velocity of cracking. A first step on this way will be some continuum mechanical considerations which make up the essence of this paper.

We start our representation by studying thermo-elastic materials. By making use of Gibb's fundamental equation for solids we derive the thermodynamic functions such as the internal energy, the free energy and the entropy, respectively. We introduce a new approach to the internal energy and illustrate the corresponding functions by some practical examples. We extend our representation to thermo-elasto-plastic materials and by applying the general energy balance we formulate a fracture criterion which is valid for elasto-plastic materials.

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THERMO-ELASTIC MATERIALS

We write Gibb's fundamental equation as follows

$$ds = \frac{1}{T}\left[d\varepsilon - \frac{1}{\rho_o}\,\underline{\sigma}\,d\,\underline{e}\right] \tag{1}$$

where the quantities used mean:

ρ_o: initial density of the material
T : present temperature
$\underline{\sigma}$: stress tensor
$\underline{e}$: strain tensor
s : specific entropy
ε : specific internal energy

We assume the existence and uniqueness of the following functions

$$\varepsilon = \varepsilon(\underline{e},T) \quad ; \quad s = s(\underline{e},T) \quad ; \quad \underline{\sigma} = \underline{\sigma}(\underline{e},T) \tag{2}$$

Much interesting and helpful work has been done in the area of thermo-elasticity by Müller (1), Mazilu (2), Becker and Bürger (3) and Bui et al. (4). So the reader should compare the results of the present paper with those of the authors cited above. Mainly we agree with (1) and our basic assumptions are in agreement with this theory.

We substitute the total differential in equation (1) and obtain

$$ds = \frac{1}{T}\left[\frac{\partial\varepsilon}{\partial T}\bigg|_{\underline{e}} dT + \left(\frac{\partial\varepsilon}{\partial\underline{e}}\bigg|_{T} - \frac{1}{\rho_o}\,\underline{\sigma}\right) d\underline{e}\,\right] \tag{3}$$

and

$$\underline{\sigma} = \rho_o\,\frac{\partial(\varepsilon - Ts)}{\partial\underline{e}} \tag{4}$$

which is a corollary of equation (3).

Thereby the term within the parenthesis is the definition of the free energy $f = \varepsilon - Ts$. From equation (4) it follows that

$$\frac{\partial\varepsilon}{\partial\underline{e}}\bigg|_{T} = \frac{1}{\rho_o}\,\underline{\sigma} + T\,\frac{\partial s}{\partial\underline{e}}\bigg|_{T} \tag{5}$$

is valid. By forming the differential of the specific free energy and together with equation (1) the following relation holds

$$\left.\frac{\partial s}{\partial \underline{e}}\right|_T = -\frac{1}{\rho_o}\left.\frac{\partial \underline{\sigma}}{\partial T}\right|_{\underline{e}} \tag{6}$$

Further, the constitutive equations of linear thermoelasticity are used

$$\sigma_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu} e_{kk}\delta_{ij} \right) - \frac{E\alpha}{1-2\nu}\,\Theta\delta_{ij} \tag{7}$$

where the quantities used mean:

E : Young's modulus
ν : Poisson's ratio
T_o: reference temperature
α : linear coefficient of thermal expansion
Θ : temperature differential ($\Theta = T - T_o$)

Thus, by using the new approach to the internal energy

$$\left.\frac{\partial \varepsilon}{\partial T}\right|_{\underline{e}} = c_o + c_1\,\Theta \tag{8}$$

as well as the relations (5), (6) and (7) it can be shown that the expression for the internal energy reads as follows

$$\varepsilon = w_e + \frac{1}{\rho_o}\,\frac{E\alpha}{1-2\nu}\,T_o e_{kk} + c_o T + \frac{1}{2}\,c_1\Theta^2 \tag{9}$$

where w_e is the strain energy density according to

$$w_e = \frac{E}{2(1+\nu)}\left(e_{ij}e_{ij} + \frac{\nu}{1-2\nu}\,e_{kk}^2\right) \tag{10}$$

From the preceding equations the expressions for the free energy and the entropy, respectively, can be derived

$$f = w_e - \frac{1}{\rho_o}\,\frac{E\alpha}{1-2\nu}\,\Theta e_{kk} - c_o T\ell n\left(\frac{T}{T_o}\right) - \frac{1}{2}c_1\Theta^2 + c_1 T_o\left(T\ell n\left(\frac{T}{T_o}\right) - \Theta\right) \tag{11}$$

$$s = \frac{1}{\rho_o}\,\frac{E\alpha}{1-2\nu}\,e_{kk} + c_o\left(1 + \ell n\left(\frac{T}{T_o}\right)\right) + c_1\left(\Theta - T_o\ell n\left(\frac{T}{T_o}\right)\right) \tag{12}$$

The quantities c_o and c_1 are constants. If c_1 equals zero, then c_o is the well known specific heat at constant deformation. The second constant c_1 gives a better approximation and has a clear physical meaning which can be shown in the following manner.

Firstly, we restrict our considerations to processes which are locally adiabatic. With regard to elasto-plastic materials this is no disadvantage because the non-stationary heat conduction problem can be solved as exactly as necessary - see U. Blix (5). Further, by using the result of a heat energy balance for the adiabatic case without any sources

$$-\frac{1}{\rho_o}\frac{E\alpha}{1-2\nu}e_{kk} = c_o \ell n\left(\frac{T}{T_o}\right) + c_1\left(\Theta - T_o \ell n\left(\frac{T}{T_o}\right)\right) \tag{13}$$

the expressions for the internal and free energy are obtainable from the equations (9) and (11), respectively, as follows

$$\varepsilon = w_e + c_o T_o\left[1+\frac{\Theta}{T_o} - \ell n\left(1+\frac{\Theta}{T_o}\right)\right] + c_1 T_o^2\left[-\frac{\Theta}{T_o}+\frac{1}{2}\left(\frac{\Theta}{T_o}\right)^2 + \ell n\left(1+\frac{\Theta}{T_o}\right)\right] \tag{14}$$

$$f = w_e - c_o T_o \ell n\left(1+\frac{\Theta}{T_o}\right) + c_1 T_o^2\left[-\frac{\Theta}{T_o}+\frac{1}{2}\left(\frac{\Theta}{T_o}\right)^2 + \ell n\left(1+\frac{\Theta}{T_o}\right)\right] \tag{15}$$

Then, from equation (12) the constant c_o for the entropy reads $s = c_o$. Now, we give an interpretation of the constant c_1 and consider a cooling process for a body with free thermal expansion. The process runs from T_o to $T = 0$ (absolute zero). With equation (9) and $\Theta = -T_o$ we write the internal energy as follows

$$\varepsilon(-3\alpha T_o, 0) = -\frac{3}{2}\frac{E\alpha^2}{\rho_o(1-2\nu)}T_o^2 + \frac{1}{2}c_1 T_o^2 \tag{16}$$

It is admissible to put $\varepsilon(-3\alpha T_o, 0)$ equal to zero and from equation (16) it follows then

$$c_1 = 3\frac{E\alpha^2}{\rho_o(1-2\nu)} \tag{17}$$

By carrying out a second experiment where a body with constant strain is cooled down to a temperature $T = 0$ the equations (9) and (17), respectively, lead to

$$\varepsilon(0,0) = \frac{3}{2}\frac{E\alpha^2}{\rho_o(1-2\nu)}T_o^2 \tag{18}$$

Finally, from equation (11) the expression for the free energy is obtainable. In conclusion it can be stated that the constant c_1 is necessary to guarantee the continuity of the thermodynamic function near the temperature $T = 0$. Moreover, it is obvious that c_1 is an additional constant which is necessary for getting a thermodynamic function with a quadratic temperature dependence. In order to demonstrate the behaviour of the thermodynamic functions some typical processes have been calculated in the figures (1) - (4).

ELASTO-PLASTIC MATERIALS

Continuum Mechanical Foundation

Firstly, we make a reference to the excellent papers of Buggisch et al (6) and Lehmann (7). Many assumptions and theorems used in this paper have been described by these authors in detail. So we can restrict our representations to some essential equations which are necessary to formulate the J-integral in a vectorial notation.

The general energy balance in its local and integral form, respectively, reads

$$\rho_o \dot{\phi} = \psi_{j,j} + \dot{\kappa} \tag{19}$$

$$\frac{d}{dt} \int_B \rho_o \phi \, dV = \int_{\partial B} \psi_j n_j \, dA + \int_B \dot{\kappa} \, dV \tag{20}$$

where the quantities used mean:

ϕ : stored energy
$\psi_{j,j}$: energy flux
κ : energy production of internal sources

These energy terms can be subdivided as follows

$$\dot{\psi} = \dot{\varepsilon} + \frac{1}{2} (\dot{u}_k \dot{u}_k)^{\bullet} + \dot{w}_s + \dot{w}_o + \dot{w}_\beta \tag{21}$$

$$\dot{\kappa} = \rho_o k_i \dot{u}_i + \dot{R} \tag{22}$$

$$\psi_j = \sigma_{ij} \dot{u}_i - q_j \tag{23}$$

where the quantities used mean:

u_k : displacement
w_s : energy stored in the structure
w_o : surface energy
w_β : energy stored in the structure of the fracture zone
k_i : body force
R : heat source
q_j : heat flux vector
w_γ : dissipated fracture energy

The following considerations are restricted to elasto-plastic materials with a quasi-static behaviour so that $k_j = 0$, $\dot{R} = 0$ and the change of the kinetic energy equals zero. By using the result of a heat energy balance it follows from equation (9) for the internal energy rate

$$\dot{\varepsilon} = \dot{w}_e - \frac{1}{\rho_o}\frac{E\alpha}{1-2\nu}\Theta\dot{e}_{ii} + \dot{w}_D + \dot{w}_\gamma - \frac{1}{\rho_o} q_{j,j} \tag{24}$$

with w_D as the energy dissipation outside of the fracture zone. Thus, by applying the equations (21) - (24) the energy balance (20) reads

$$\int_{\partial B} [(\rho_o w_e \delta_{jk} - \sigma_{ij} u_{i,k})^{\bullet} + (-\frac{E\alpha}{1-2\nu}\Theta\dot{e}_{ii} + \rho_o\dot{w}_I + \rho_o\dot{w}_B)\delta_{jk} + \\ + \dot{\sigma}_{ij}u_{i,k} - \sigma_{ij,k}\dot{u}_i] dA_j = 0 \tag{25}$$

where $w_I = w_D + w_S$ means the irreversible energy outside of the fracture zone and $w_B = w_O + w_\beta + w_\gamma$ is the fracture energy. Moreover, the Gauß-Green theorem has been used for the transformation of several surface integrals. Further, it is required that the surface line ∂B is situated exclusively within the linear-elastic part of the body. After an integration with respect to time it follows from equation (25)

$$\int_{\partial B} [(\rho_o w_e - \frac{E\alpha}{1-2\nu}\int\Theta de_{ii})\delta_{jk} - \sigma_{ij}u_{i,k}] dA_j + \int_B \rho_o (w_{I,k} + w_{B,k})\, dV = 0 \tag{26}$$

which can be considered as the conservation law for elasto-plastic material under the assumption that all energy fields have zero values at the beginning of the process.

Application of the Conservation Law to Fracture Processes

Firstly we should mention the very interesting results of Gurtin [8], [9] who has given a derivation of the J-integral on a thermodynamic basis.

Designating the first integral in equation (26) by the quantity J_k the equation (26) reads as follows

$$\int_B \rho_o (w_{I,k} + w_{B,k})\, dV = - J_k \tag{27}$$

It was shown by Kaempf and Herrmann (10) that the so defined J_k-integral is path-independent like the well-known J-integral originated by Rice (11). Calling the length of the crack path by a the equation (27) can be reformulated as follows

$$J_k = - \int_B \rho_o (w_{I,a} + w_{B,a})\, a_{,k}\, dV \tag{28}$$

It can be shown that with $a_{,k}$ = constant the fracture criterion may be formulated as follows

$$W_{I,a} + W_{B,a} \leqq - J_k \frac{\partial x_k}{\partial a} \tag{29}$$

where $W_{I,a}$ and $W_{B,a}$ are the increasing irreversible and fracture energies, respectively, with respect to an increasing fracture path.

REFERENCES

(1) Müller, I., "Thermodynamic Theories of Thermoelasticity", in: "The Constitutive Law in Thermoplasticity". Edited by Th. Lehmann, CISM Course and Lectures No. 281, Springer 1984, pp.13-104.

(2) Mazilu, P., Variationsprinzipe der Thermoplastizität II. Gekoppelte thermodynamische Prozesse, Mitt. aus dem Institut für Mechanik, No.37, Universität Bochum, West Germany, 1983.

(3) Becker, E. and Bürger, W., "Kontinuumsmechanik", Teubner Studienbücher Mechanik, Stuttgart, West Germany, 1975.

(4) Bui, H.D., Ehrlacher, A., Nguyen, Q.S. Journal de Mécanique, Vol.19, No.4, 1980, pp.697-723.

(5) Blix, U., Zur Berechnung der Einschnürung von Zugstäben unter Berücksichtigung thermischer Einflüsse mit Hilfe der Finite-Element-Methode, Mitt. aus dem Institut für Mechanik, No.40, Universität Bochum, West Germany, 1983.

(6) Buggisch, H., Gross, D. and Krüger, K.-H., Ingenieur-Archiv No.50, 1981, pp.103-111.

(7) Lehmann, Th., Große elasto-plastische Formänderungen, Mitt. aus dem Institut für Mechanik No.1, Universität Bochum, West Germany, 1976.

(8) Gurtin, M.E., On the energy release rate in quasi-static elastic crack propagation, Journal of Elasticity, Vol.9, 1979, pp.187-195.

(9) Gurtin, M.E., Thermodynamics and the Griffith criterion for brittle fracture, Int. J. Solids Structures, Vol.15, 1979, pp.553-560.

(10) Kaempf, B, Herrmann, K.P., Some remarks about the J-integral for thermo-elastic materials, Int. J. of Fracture, to be published, 1986.

(11) Rice, J.R., "Mathematical Analysis in the Mechanics of Fracture", in: "Fracture. An Advanced Treatise". Edited by H. Liebowitz, Vol.II, Academic Press 1968, pp.191-311.

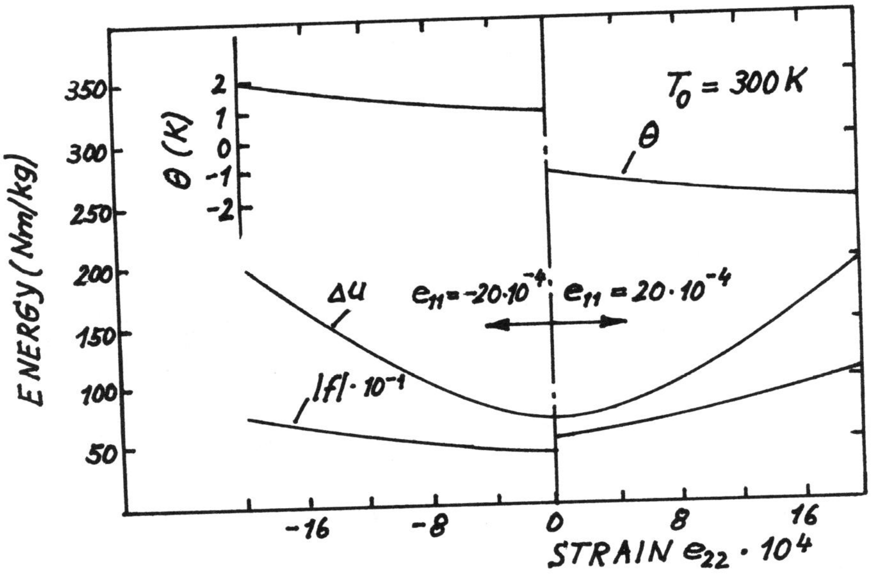

Figure 1 Free energy f, increment of internal energy Δu and temperature difference Θ for plane state of strain

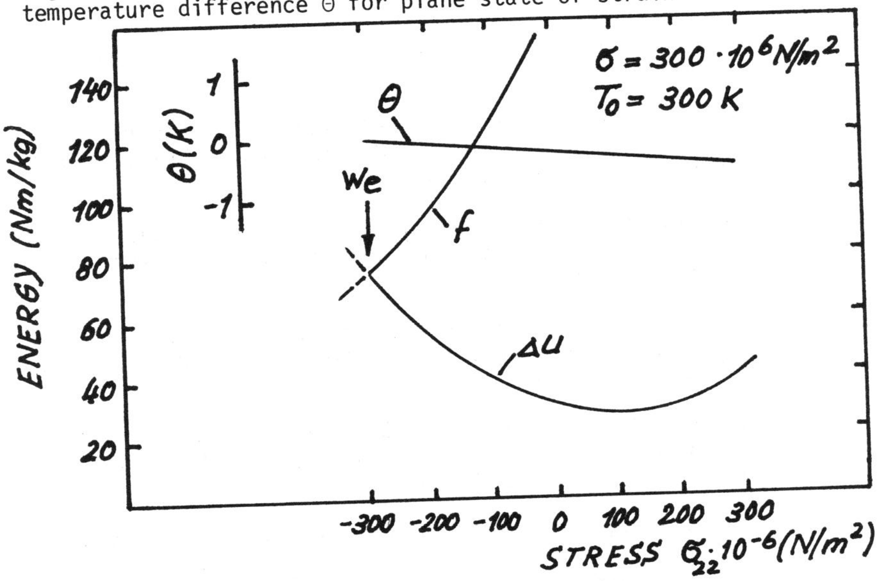

Figure 2 Free energy f, increment of internal energy Δu and temperature difference Θ for plane state of stress

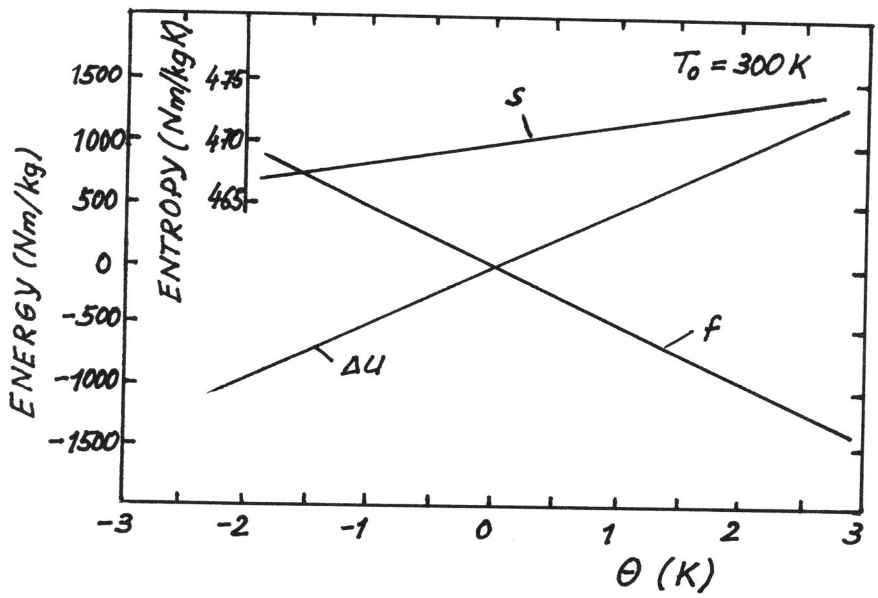

Figure 3 Free energy f, increment of internal energy Δu and entropy s for a stress free body

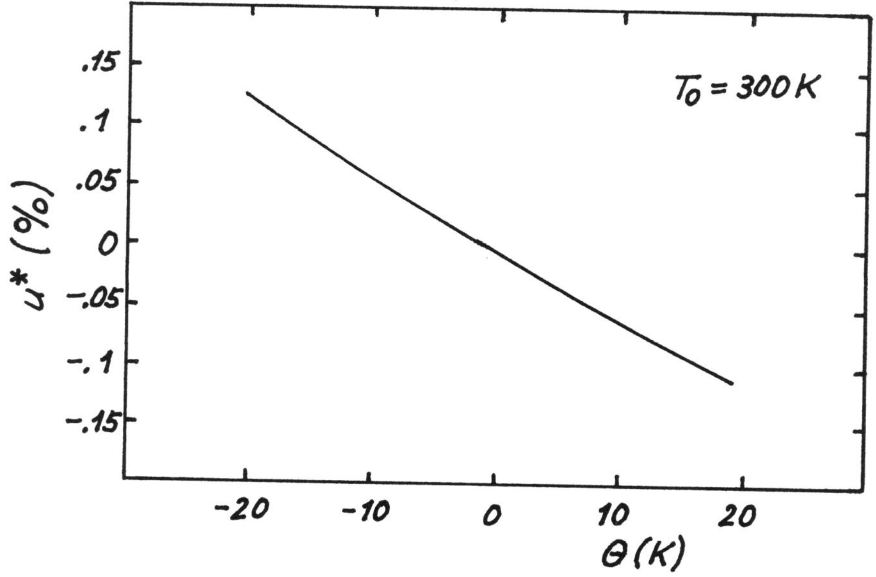

Figure 4 Ratio u* between the difference of the internal energies of a clamped and a stress-free body, respectively, related to the stress-free state