AN STATISTICAL APPROACH TO FATIGUE CRACK GROWTH UNDER RANDOM LOADING

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A method to calculate the fatigue crack growth under wide band random loading is presented. In this paper the basic data are the probability density function (p.d.f.) of stress and one fatigue crack growth rate equation. The approach is based on a statistical representation of the increase of crack length at any cycle as a function of the previous length and the p.d.f. of stress range. A recursive application of this approach will produce an expected value of the number of cycles to produce failure.

INTRODUCTION

The process of fatigue is usually divided, as far as the effects of calculation are concerned, into the periods of initiation and propagation of cracks. In some cases the propagation occupies the greatest part of the fatigue process. It is evident, therefore, that the development of a sufficiently precise method of calculation is required.

At the present time, the most frequently employed method in the analysis of the propagation period is based upon Linear Elastic Fracture Mechanics (LEFM). In the case of loads of a constant amplitude, the duration of the propagation period is determined by directly integrating an equation that relates the crack growth rate (da/dN) to the loads. Forman's equation (1) is an example of this.

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When the loads vary in an irregular or random form the life is usually calculated by means of cycle by cycle simulation. This process however requires a great deal of time. To solve this problem, Barsom (2) and Hudson (3) relate the values of the root-mean-square stress and stress intensity factor (ΔS_{RMS} and ΔK_{RMS}) produced during the loading process to the growth rate in a similar way as in the case of constant amplitude. An estimation of the duration of the propagation process can be obtained by directly integrating the equation which relates the crack growth rate with ΔS_{RMS} or ΔK_{RMS}.

Presented in this paper is a statistical approximation of this process based upon the distribution of the stress range in the load history.

DESCRIPTION OF THE MODEL

The model here presented is based upon the determination of the expected length of a crack after each cycle of load. When the crack length reaches the critical value or a previously defined value it can be assumed that the propagation has been finalized.

Beginning with an initial length (a_o), the length after any cycle n can be expressed as:

$$a_n = a_{n-1} + \Delta a_n \tag{1}$$

where a_{n-1} is the length of the crack after n-1 cycles and Δa_n is the increase in the length produced in cycle n. The expected value of a_n can be written as:

$$E\left[a_n\right] = E\left[a_{n-1} - \Delta a_n\right] \tag{2}$$

This equation rewritten can be expressed as:

$$E[a_n] = \int_{a_o}^{\infty} \int_{o}^{\infty} (a_{n-1} + \Delta a_n)\, p(a_{n-1}, \Delta a_n)\, da_{n-1}\, d\Delta a_n \tag{3}$$

where $p(a_{n-1}, \Delta a_n)$ is the joint probability density function, which can also be written:

$$p(\Delta a_n, a_{n-1}) = p(\Delta a_n / a_{n-1}) \cdot p(a_{n-1}) \tag{4}$$

where $p(\Delta a_n, a_{n-1})$ is the conditional probability density function.

Substituting equation (4) into equation (3) will give :

$$E[a_n] = E[a_{n-1}] + \int_{a_o}^{\infty} \int_{o}^{\infty} \Delta a_n \, p(\Delta a_n/a_{n-1}) \cdot p(a_{n-1}) da_{n-1} \, d\Delta a_n \tag{5}$$

Assuming in $p(\Delta a_n/a_{n-1})$ that $a_{n-1} = E[a_{n-1}]$, we have after some calculations:

$$E[a_n] = E[a_{n-1}] + \int_{o}^{\infty} \Delta a_n \, p(\Delta a_n/E[a_{n-1}]) \, d\Delta a_n \tag{6}$$

Equation (6) is a recursive formula to determine the expected length of the crack in each cycle. In order to evaluate this equation, some expression of Δa_n and $p(\Delta a_n/E[a_{n-1}])$ is necessary. To express Δa_n we can use any one of the crack growth rate equations because Δa_n is the increment of the length of the crack in each cycle, that is $\Delta a_n = da/dN$. In all of the models of crack growth based upon Linear Elastic Fracture Mechanics, da/dN is a function of ΔK and therefore of ΔS and a. Given this, we can write:

$$\Delta a_n = f\,(\Delta s, a_{n-1}) = f\,(\Delta s, E[a_{n-1}]) \tag{7}$$

The probability that Δa_n is in the region Δa_n to $\Delta a_n + d\Delta a_n$ is equal to the probability that the range of stress in the n^{th} cycle, which produces the increase in the length Δa_n is in the region Δs to $\Delta s + d\Delta s$. That is:

$$p(\Delta a_n/E[a_{n-1}]) \, d\Delta a_n = p(\Delta s) \, d\Delta s \tag{8}$$

By substituting equations (7) and (8) into equation (6), we have:

$$E[a_n] = E[a_{n-1}] + \int_{o}^{\infty} f\,(\Delta s, E[a_{n-1}]) \, p(\Delta s) \, d\Delta s \tag{9}$$

The integral in this equation represents the expected value of growth per cycle when the crack has a length a_{n-1}. All of the components of the equation (9) are known, and therefore we can determine the expected value of the length of the crack at any given moment as a function of the length after the previous cycle and the statistical characteristics of the load history.

Given that the length of the crack and the rate of growth change slowly, we can approximate the growth rate equation using intervals of constant length. With a final crack length a_f and an initial crack length a_o, we can simulate the growth as a series of small increment a'_I:

$$a'_I = \frac{a_f - a_o}{H} \tag{10}$$

Where the total increase of length from the first cycle to the last cycle is divided into H increments, the length at the beginning of each interval will be: a'_o, a'_1, ...a'_{H-1}, cf. figure 1. For each increase $\Delta a'_I$ there will be a growth rate equal to the mean of the rates corresponding to the initial crack length a'_{I-1} and the final length a'_I of interval.

In accordance with the preceding equation and also with equation (9), the number of cycles necessary to obtain a crack length a_f must be:

$$N_f = \sum_{1}^{H} \frac{2\Delta a'_I}{\int_o^\infty f(\Delta s,\, a_{I-1})\, p(\Delta s)\, d\Delta s + \int_o^\infty f(\Delta s,\, a_I) p(\Delta s) d\Delta s} \tag{11}$$

in which $[\int_o^\infty f(\Delta s,\, a_{I-1})\, p(\Delta s)\, d\Delta s + \int_o^\infty f(\Delta s,\, a_I)\, p(\Delta s)\, d\Delta s]/2$ is the mean crack growth rate considered in the interval of growth from a'_{I-1} to a'_I.

In a general manner, equation (11) can be written as:

$$N_f = \int_{a_o}^{a_f} \frac{da}{\int_0^\infty f(\Delta s,\ E[a_{n-1}])\ p(\Delta s)\ d\Delta s} \qquad (12)$$

If we express the equation of growth $f(\Delta s,\ a)$, as:

$$f(\Delta s,\ a) = g(a) \cdot h(\Delta s) \qquad (13)$$

equation (12) will then be:

$$N_f = \int_{a_o}^{a_f} \frac{da}{g(a) \int_0^\infty h(\Delta s)\ p(\Delta s)\ d\Delta s} \qquad (14)$$

If we then employ, for example, the Paris' equation (4), which fulfills the conditions of equation (13), we will have:

$$g(a) = C\ (f(a))^n\ (\sqrt{\pi a})^n$$

and:

$$h(\Delta s) = \Delta s^n$$

Thus equation (14) transforms into:

$$N_f = \int_{a_o}^{a_f} \frac{da}{g(a) \int_0^\infty \Delta s^n\ p(\Delta s)\ d\Delta s} \qquad (15)$$

The integral in the denominator is the mean value of Δs^n in the history of nominal stress:

$$\int_0^\infty \Delta s^n\ p(\Delta s)\ d\Delta s = \overline{\Delta s^n} \qquad (16)$$

We can say:

$$\Delta S_{RMN} = \sqrt[n]{\overline{\Delta s^n}} \qquad (17)$$

and therefore equation (15) can be written as:

$$N_f = \int_{a_o}^{a_f} \frac{da}{g(a)\ \Delta S^n_{RMN}} \qquad (18)$$

which is a generalization of the equation proposed by Barsom (2):

$$N_f = \int_{a_f}^{a_f} \frac{da}{g(a)\ \Delta S^2_{RMS}} \qquad (19)$$

where:

$$\Delta S_{RMS} = \sqrt{\overline{\Delta S^2}} \qquad (20)$$

When the equation of growth doesn't fulfill the conditions of equation (13), the general expression is equation (12) or equation (11) which are the same.

Whatever may be the crack growth rate expression used, the proposed model requires a previous knowledge of p(Δs) and also of a_f, given that we know the initial length a_o. The value of a_f can be defined in accordance with a determined criterion or as the critical crack length after which fracture is produced. In the latter case, a_f can be determined by the equation:

$$K_C = F(a_f)\ \sqrt{\pi a_f} \int_0^\infty \Delta s\ p(\Delta s)\ d\Delta s \qquad (21)$$

where K_C is the fracture toughness and F(a) is a function of the crack length and geometry. The value of a_f obtained in this equation can be understood as the most probable value of the critical crack length.

Two different functions have been used to approximate p(Δs) from the ranges produced by the load history. One is the equation of Rice's proposal (3) and the other is a modification of this. Both of them have been displaced to the right in order to obtain a better approximation. The expressions are in order:

$$p(\Delta s) = \lambda_1 \frac{\Delta s - \Delta s_o}{\sigma^2}\ e^{-\lambda_2\ (\Delta s - \Delta s_o)^2/\sigma^2} \qquad (22)$$

$$p(\Delta s) = \lambda_1 \frac{(\Delta s-\Delta s_o)^2}{\sigma^3} e^{-\lambda_2 (\Delta s-\Delta s_o)^2/\sigma^2} \qquad (23)$$

where σ is the standard deviation of the amplitude of the history and Δs_o is the displacement of the function. The relation between the parameters λ_1 and λ_2 in both expressions can be obtained by using the condition:

$$\int_0^\infty p(\Delta s)\, d\,\Delta s = 1 \qquad (24)$$

The approximation of the functions is carried out by employing a Least-Squares method and by varying the displacement until a minimal error is obtained.

The probability density function in equation (22) which is Rice's proposal, will be called function 1 and the other function 2. The figures 2 and 3 represent approximations obtained with equations (22) and (23) respectively.

RESULTS

The application of the proposed model has been carried out by means of equation (11). The number of intervals considered is thirty. Forman's equation (4) has been used as the crack propagation rate equation f(s,a). This is:

$$\frac{da}{dN} = C \frac{\Delta K_{eff}^n}{(1-R)\,\Delta K_C - \Delta K_{eff}} \qquad (25)$$

where C and n are material parameters.
A mean value of $R = S_{max}/S_{min}$ has been used and, in order to obtain a better prediction, an effective range of the stress intensity factor (ΔK_{eff}) has been adopted. This range was obtained through the application of the crack closure concept proposed by Elber (5).

To test the proposed model, a number of cases have been studied. The results of these cases obtained with different load histories, have been compared with the results obtained in experiments, and with a limited number of simulations made by other researchers.

The load histories employed have been the same as those used by the "Fatigue Design and Evaluation Committee" of SAE (6), specifically "Brake", "Transmission" and "Suspension", and the "A-A" and "I-N" used in the "Round-Robin Crack Growth Prediction on Center Crack Tension Specimen Under Spectrum Loading" (7) which will be called "H1300" and "H300" respectively. The materials and test specimens used are those of the Round-Robin and SAE programs. The SAE program uses a modified compact tension specimen with a stress concentration factor $K_T = 3$. The materials used are MAN-TEN and RQC-100 steels. In the Round-Robin program they used a center-cracked-tension specimen of 2219-T851 aluminium alloy. Both types of steel have been used with the SAE load histories and the aluminium alloy with the other two load histories. The maximum levels of stress produced in each load history with different materials are shown in Table 1.

The crack closure stresses in the case of the aluminium alloy have been obtained from the Newman expressions (8) using a value of R as defined in equation (26). The same values as in the Round-Robin program have been used for a_o and a_f.

TABLE 1 - Maximum stress produced in each of the load histories and levels (MPa).

Material	Level	Brake	Trans.	Susp.	H300	H1300
MAN-TEN	1	176.16	402.69	453.	-	-
	2	151.16	176.16	302.	-	-
RQC-100	1	402.69	402.69	804.75	-	-
	2	176.16	-	352.	-	-
	1	-	-	-	193.	248.
2219-T851	2	-	-	-	144.79	186.
	3	-	-	-	-	124.

In Figure 4 a comparison of the results obtained in the proposed model with equations (22) and (23) and the experimental results of the Round-Robin program is shown. In this figure the results of Hudson (3) are also compared with those from experiments. In Table 2 a comparison between the results obtained by the proposed methods in (7), by our model and by the experiments is shown. The indicated values are relations between the calculated lives and the test lives. In the Round-Robin case the values are the means of all the predictions methods in (7), and the band within which they fluctuate.

As can be seen, the results obtained with these load histories and materials are fairly good. As to the effects of the probability density functions, the results are better with equation (22). Equation (23) always predicts a longer life for the same load history than with the other function. At any rate, all of the results are within a band ranging from 1/2 to 2, when related to the test results.

Figures 5 and 6 represent the results obtained with the three SAE load histories compared with those of the tests. The crack propagation in this case is between a crack length of 2.5 mm. and the failure crack length a_f. Regarding the two steels used, the crack propagation threshold value adopted has been:

$$\Delta K_{th} = 11 \text{ MPa} \sqrt{m}$$

The value of the crack closure stress has been assumed to be equal to 35% of the maximum in all cases, as proposed by Nelson (9).

With the proposed model, relatively good results have been obtained except in the case of the "Suspension" history, especially at high levels. This history has a mean value of R which is much less than minus 1 ($R \ll -1$). This implies smaller values of the closure stress than in other histories; however, they are considered to be of the same value. Furthermore, in the case of high levels of stress the mean of the minimal nominal stress proves to be half of the yield stress, with numerous cycles that surpass this yield stress. This, besides affecting the closure stress even more, exceeds the limits of applicability of Linear Elastic Fracture Mechanics.

In as much as the expressions used to approximate the probability density function are concerned, both equations produce very similar results. Anyway, the results corresponding to equation (22) are in more cases a little better than those of equation (23).

TABLE 2 - Comparison between Results in (7) and those obtained by the proposed Method (Calculated Life/Test Life), using Equations (22) and (23)(Functions 1 and 2 respectively).

		level (table 1)		
History	Case	1	2	3
H 300	Round-Robin	1.20±.41	1.60±.64	-
	Function 1	1.29	1.60	-
	Function 2	1.45	1.80	-
H1300	Round-Robin	1.03±.30	.99±.21	1.47±.43
	Function 1	1.03	1.14	1.24
	Function 2	1.32	1.36	1.41

CONCLUSIONS

Upon examining the results obtained, it can be stated that, when the sequence effects are not very strong the proposed model can be employed to rapidly determine the duration of the crack propagation period as efficiently as other methods of simulation.

The greatest advantage of the proposed model is that its rapidity does not result in miscalculation. The time of the calculation is constantly maintained independent of the life and of the number of cycles in the load history. This permits us to use larger histories which characterize the variations of the loads in a more exact manner.

The principal inconvenience of the proposed model is the dificulty to consider, at least in a direct manner the effects of the sequence.

SYMBOLS USED

a_o	= initial crack length (m)
a_f	= final crack length (m)
a_n	= crack length after cycle number n (m)
da/dN	= crack growth rate (m/cycle)
$E[a_n]$	= expected value of a_n (m)
K_c	= critical value of K to fracture (MPa $\sqrt{m}$)
K_T	= elastic stress concentration factor
N_f	= number of cycles to failure
p(x)	= probability density function (p.d.f.) of x
R	= cycle ratio K_{min}/K_{max}
$\bar{x}$	= mean value of x
x_{RMS}	= root mean square value of x
Δa_n	= increase of length produced by cycle n (m)
ΔK	= range of stress intensity factor (MPa $\sqrt{m}$)
ΔS	= range of nominal stress (MPa $\sqrt{m}$)
λ_1, λ_2	= parameters of the probability density function of ranges
σ	= standard deviation

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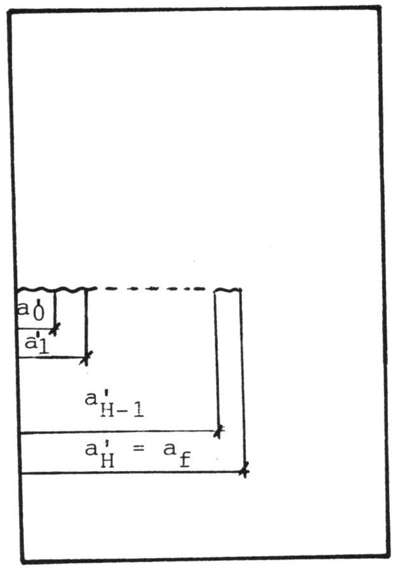

Figure 1 Division of the total growth of the crack into H increments

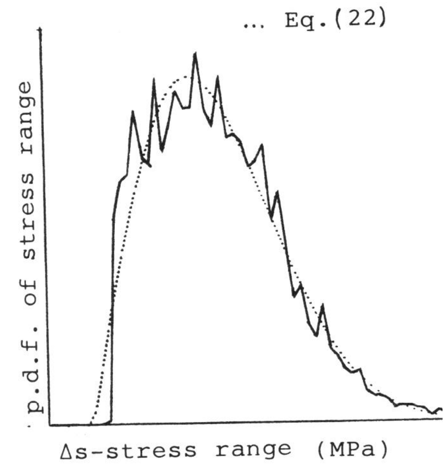

Figure 2 p.d.f. of stress range approximated by eq. (22).

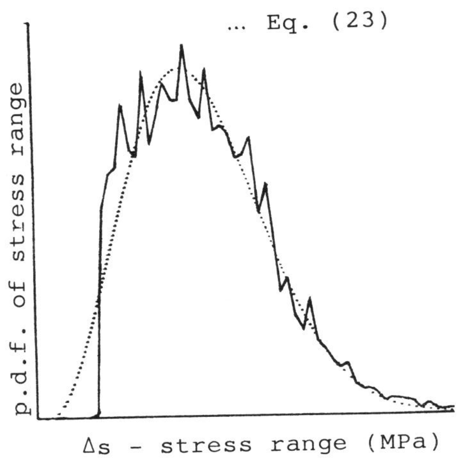

Figure 3 p.d.f. of stress range approximated by eq. (23).

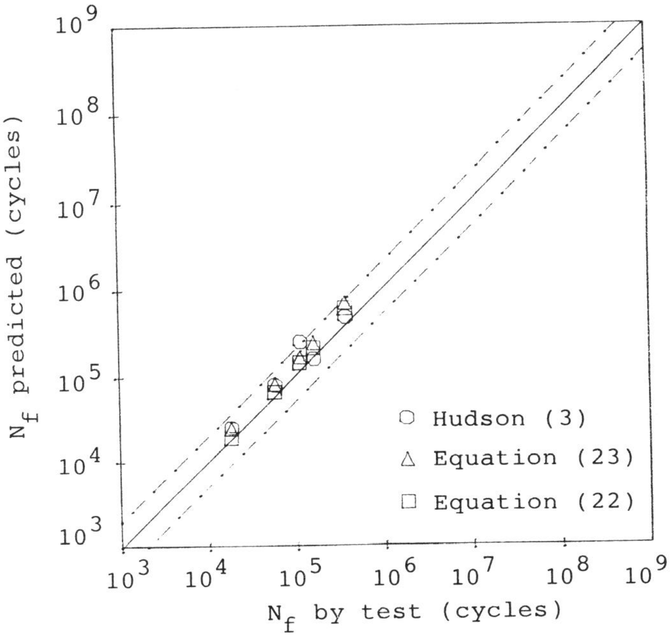

Figure 4 Comparison of experimental (8) and predicted cycles to failure for 2219-T85 aluminum alloy.

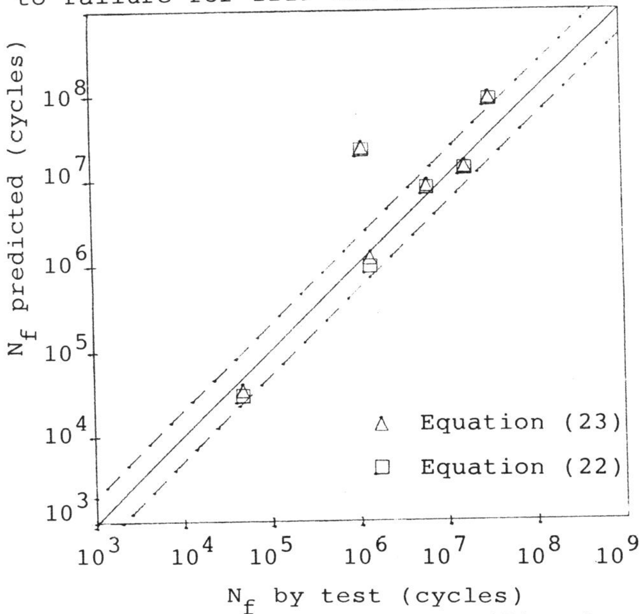

Figure 5 Comparison of experimental (7) and predicted cycle to failure for MAN-TEN.

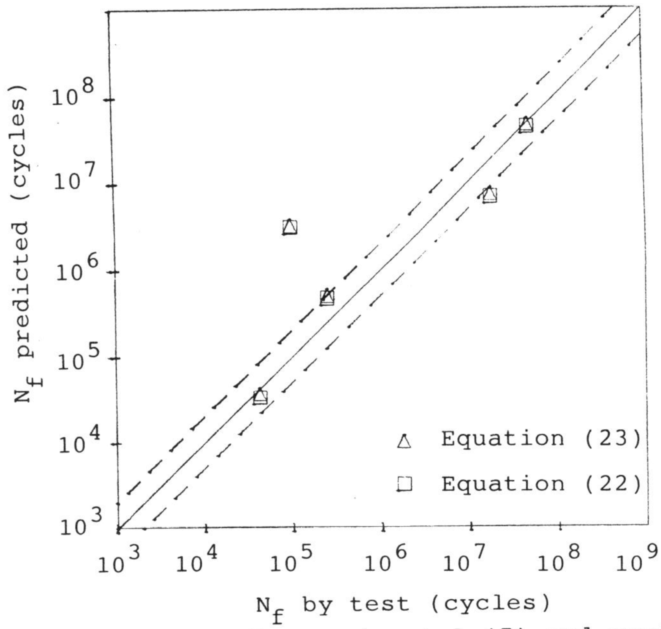

Figure 6 Comparison of experimental (7) and predicted cuclyes to failure for RQC-100.