A RELIABILITY ANALYSIS OF FATIGUE CRACK GROWTH UNDER RANDOM LOADING AND LIFE PREDICTION

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Based on a fracture mechanics approach, a mathematical model for predicting the reliability of components subjected to random fatigue loading has been developed by applying probabilistic techniques. The analysis includes the influence of the loading and material variance, geometry and crack shape. The model has been applied to predict the distributions of crack length and fatigue life, the theoretical calculations have been shown to be in good agreement with the results of Monte-Carlo simulations and experiments

INTRODUCTION

Fatigue life prediction is an essential requirement in engineering calculations in which the fracture mechanics approach often depends on the Paris Law (1) which relates the crack growth rate $^{da}/_{dN}$ to the stress intensity range ΔK

$$\frac{da}{dN} = \alpha(\Delta K)^n \qquad \text{..........(1)}$$

where both α and n are material constants which can be determined by experiment using sinusoidal loading and conventional specimens. To assess the fatigue life of componenets under random loading, variability of the loading and material strength, geometry of the specimen and crack, should be considered. Random loadings are classified into broad and narrow band according to the width of their spectral density, see for example Fig 1. For narrow band loading, the probability distribution function of the peaks x is known to be the Rayleigh distribution

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$$p(x) = \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right) \quad \text{........(2)}$$

σ is the r.m.s of the signal. The PDF of the ranges S is then

$$p(S) = \frac{S}{4\sigma^2} \exp\left(\frac{-S^2}{8\sigma^2}\right) \quad \text{........(3)}$$

For broad band loading, cycle counting methods (for instance, Matsuishi and Endo (2), Teichmann (3)) can be used to found a series of cycles of same fatigue effects. It has been suggested (Hancock and Gall (4)) that the PDF of S in non-dimensional form can be expressed as

$$p(S/\sigma) = \frac{1}{\gamma m} \left(\frac{S}{\sigma}\right)^{m-1} \exp\left[-\left(\frac{S}{\gamma\sigma}\right)^m\right] \quad \text{........(4)}$$

where both γ and m depend upon the spectral density W(f). In this work, the variability in material strength is considered by treating α in the Paris law as a random variable. The theoretical model has been applied to predict the distributions of crack length, cycles to grow a crack to a specified length and the component life, the results are compared with Monte-Carlo simulations and experiments.

THE MEAN CRACK GROWTH RATE

The expression for the stress intensity range has a general form

$$\Delta K = f\, S \sqrt{(\pi a)} \quad \text{........(5)}$$

where f is a function of the geometry. Substituting this into Eq 1, the non-dimensional form of the Paris law becomes

$$\frac{d(a/a_0)}{dN} = \lambda\, f^n\, \pi^{n/2} \left(\frac{S}{\sigma}\right)^n \left(\frac{a}{a_0}\right)^{n/2} \quad \text{......(6)}$$

For convenience, let

$$\lambda = \alpha\, \sigma^n\, a_0^{\frac{n-2}{2}} \quad \text{.........(7)}$$

where a_0 is the initial crack length. Since the applied range is uncertain, the crack growth rate at length a is a random variable whose mean value is

$$E\left[\frac{da}{dN}\right] = \alpha\left[f(\sqrt{\pi a})\right]^n \int_0^\infty S^n\, p(S)\, dS$$

$$= \alpha\left[f\sqrt{(\pi a)}\right] \mu_n \quad \text{................(8)}$$

Given that μ_n is the mean of S^n, then for narrow band loading

$$\mu_n = E[S^n] = (8\,\sigma^2)^{n/2}\,\Gamma\left(\frac{n}{2} - 1\right) \quad \ldots\ldots\ldots(9)$$

where $\Gamma(\ )$ is the gamma function. For broad band loading

$$\mu_n = E[S^n] = (\gamma\sigma)^n\,\Gamma\left(\frac{n}{m} - 1\right) \quad \ldots\ldots\ldots(10)$$

As an example, Fig 2 illustrates the crack length against the number of cycles which would result from a central crack in an infinite plate growing at the average rate and the average plus and minus one and two standard deviations. The equation for the central line is

$$\frac{a^*}{a_o} = \left[\frac{2 - n}{2}\,\lambda\,(8\pi)^{n/2}\,\Gamma\left(\frac{n}{2}-1\right)N - 1\right]^{\frac{2}{2-n}} \quad \ldots\ldots(11)$$

DISTRIBUTIONS OF CRACK LENGTH AND LIFE

The crack length is given by summing the increments of crack on each cycle. Eq 11 can be rearranged to give

$$\int_1^{a_f/a_o} \frac{d(a/a_o)}{f^n\,\pi^{n/2}(a/a_o)^{n/2}} = \lambda \sum_{i=1}^{N} \left(\frac{S_i^n}{\sigma}\right) \quad \ldots\ldots(12)$$

where a_f is the final crack lenth and N is the number of cycles. The summation in the right hand side of this equation is a random variable, whose probability distribution can be described by the central limit theorem, therefore the left hand side is also a random variable which is denoted as X

$$X = \int_1^{a_f/a_o} \frac{d(a/a_o)}{f^n\,\pi^{n/2}(a/a_o)^{n/2}} \quad \ldots\ldots\ldots(13)$$

The distribution of X determines the distribution of a_f/a_o which is expressed as

$$p_3(a_f/a_o) = \frac{\sigma^n\, e^{-\left[\frac{X}{\alpha\, a_f^{\frac{n-2}{2}}} - N\mu_n\right]^2 / [2N\sigma_n^2]}}{\sigma_n\,\sqrt{(2\pi N)}}\left|\frac{dX}{d(a_f/a_o)}\right| \quad \ldots(14)$$

Following a similar argument, the distribution of cycles to grow a crack to a fixed length is found

$$p(N) = \frac{X \sigma^n}{\sqrt{2\pi}\, \sigma_n \lambda N^{3/2}} \exp\left[\frac{\mu_n^2 \left(\frac{X \sigma^n}{\lambda \mu_n} - N \right)^2}{2 \sigma_n^2 N} \right] \quad \ldots(15)$$

It can be shown that the central part of p(N), i.e when N is not allowed to differ too much from its mean, is approximately a normal distribution with the mean of N as

$$\mu_N = \frac{X \sigma^n}{\lambda \mu_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Eqs 14 and 15 are applicable for both narrow and broad band loading. In order to illustrate the applicance of the model, Monte-Carlo simulations and theoretical results for cumulative distribution are presented in Fig 3, 4, 7 and 8. In Fig 5 and 6, the effect of λ and n on μ_N has been shown.

MATERIAL VARIANCE AND EXPERIMENTAL COMPARISON

By treating one of the material parameters, α as a random variable, the effect of variance in material strength has been introducedinto the model (Huang (5)). This enables the comparison to be extended to experiments satisfying the conditions laid down during the deduction, for instance the experiment by Talreja (6). In his experiments, narrow band random loading of σ = 182.4 MPa and mean stress S_m = 167 MPa was applied to Cr-Mo-V steel bars of cross section 10×15 mm dimensions containing an initial crack of 0.05 mm deep on one of its 15mm faces. The data are presented in Table 1 in terms of period (each 200×12.5 cycles). Since Group 2, 3 and 4 are not comparable, they are omitted.

TABLE 1 – Eperimental Data from Talreja's Paper
Group 5 and 6 Were Tested for 150 and 200 Periods
"*" Indicates " the Rest of the Group Unfailed"

Group No.	Specimen No. 1	2	3	4	5	6	7	8	9	10	11	12
1 (10 specimens)	95	101	144	145	160	163	171	220	235	296		
5 (30 specimens)	137	149	*									
6 (18 specimens)	132	136	151	152	159	159	176	180	184	188	199	*

To establish a base for the comparison, it is assumed that n in the Paris Law is 3 and the shape of the crack follows a preferred route (Scott and Thorpe (7)), so that stress intensity correction factor f for the deepest point is evaluated accordingly (Holdbrook and Dover (8)). The definition for failure is when the crack penitrates three quarters of the material thickness, the input of μ_α = 1.15×10 and σ_α = 2.76×10 into Eq 43 produces good curve fitting to the data of Group 1 as shown in Fig 9. Using the defined parameter values, from the theoretical model, survival probabilities after 150 and 200 periods of loading are found to be 72.48% and 35.80% which are in good agreement with the experiment data of 93.3% and 38.9% respectively

CONCLUSIONS

An efficient method for assessing fatigue life has been demonstrated. The relation between data obtained from conventional tests and from fatigue tests on components has been found and checked against experiment.

The mean life of components is nearly a linear function of λ, which is a parameter consisting of α, initial crack length and the stress amplitude, while it is much more sensitive to the exponent n in the Paris law when n is small rather than large. Any increase in either λ or n causes a decrease in the mean life.

In case of significant interaction between cycles, this method may not produce accurate predictions without proper modification. Moreover, the present failure condition is when the crack propagates to a critical size, which is appropriate because of the large growth rate close to fracture of the component. To include these effect into the model, more research is needed.

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SYMBOLS USED

a_f = final crack length

a_0 = initial crack length

a = crack length

a* = length of the crack with average growth rate

E = the mean of

f = geometrical calibration factor

m = a parameter depending on W(f)

N = number of cycle

n = exponent in Paris law

p_i = probability density function

S = stress range

W = power spectrum of the signal

X = a value defined by Eq 13

x = peak of narrow band signal

α = factor in Paris law

γ = a parameter depending on W(f)

σ = the r.m.s. of the signal

λ = a facter defined by Eq 7

$\Gamma(\ )$ = Gamma function

μ_n = average of S^n

σ_n = r.m.s. of S^n

μ_N = verage of cycle numbers

μ_α = mean of α

σ_α = r.m.s of α

ΔK = stress intensity range

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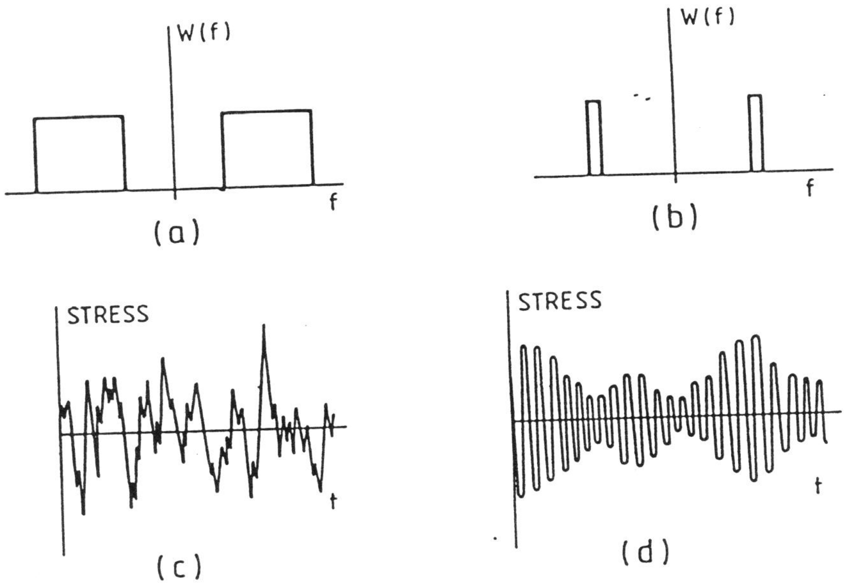


Fig 1 a and c: Broad band spectrum and signal
b and d: Narrow band spectrum and signal

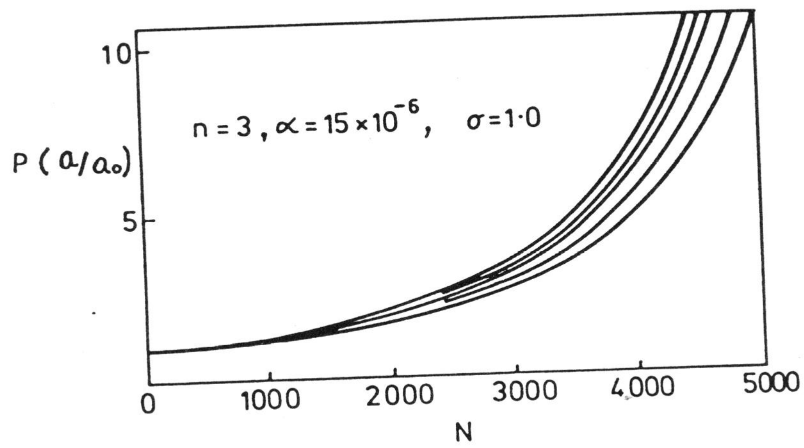
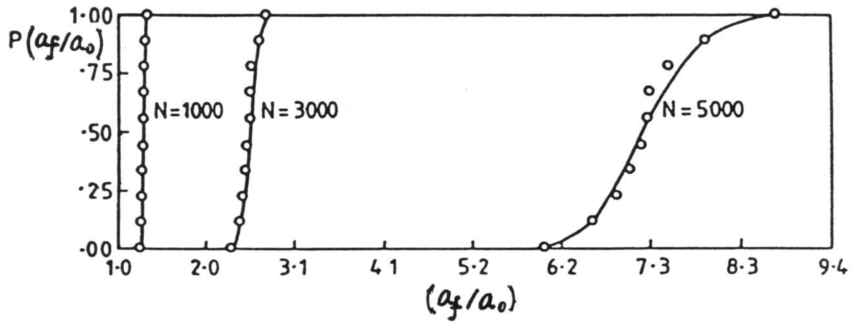
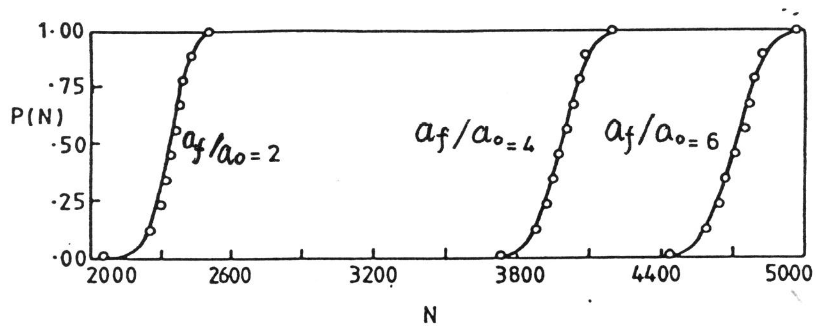


Fig 2 The difference between cracks with different growth rate



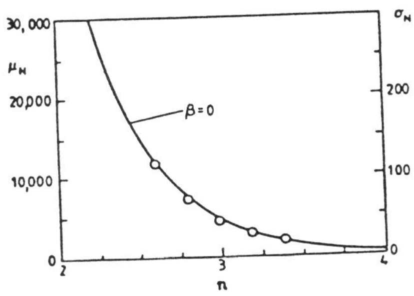
$\lambda = 1.5E\text{-}6,\ n = 3,\ f = 1$

Fig 3 The distribution of crack length under narrow band loading "——" result from Eq 24, " o " Monte-Carlo simulation result



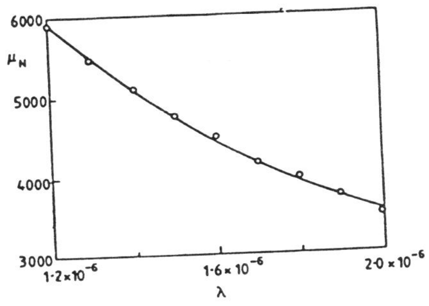
$\lambda = 1.5E\text{-}6,\ n = 3,\ f = 1$

Fig 4 The distribution of cycle numbers under narrow band loading "——" result from Eq 29, " o " Monte-Carlo simulation result



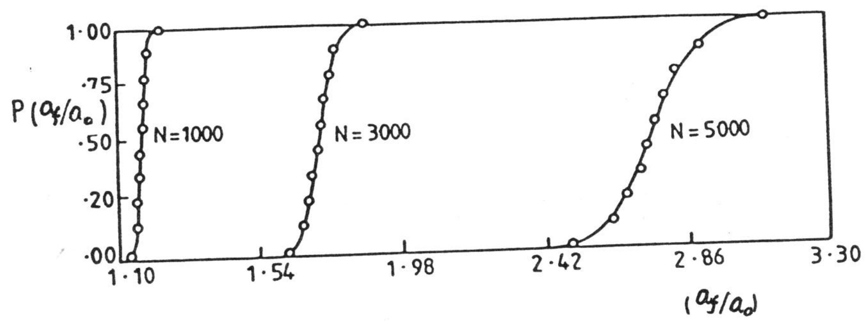
$a_f/a_0 = 6 \quad \alpha = 1.5E-6$
$f = 1$

Fig 5 The effect of n on μ_n, narrow band loading



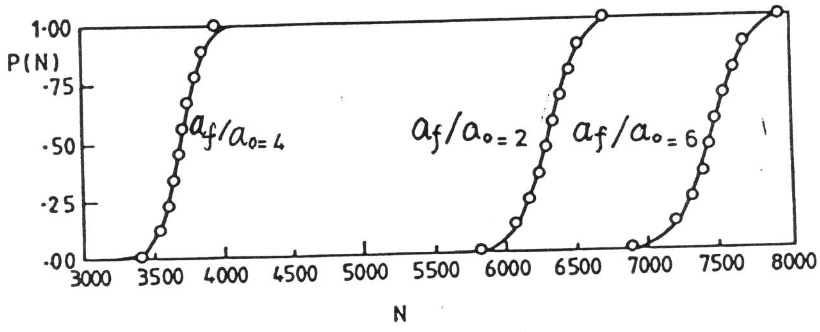
$a_f/a_0 = 6 \quad n = 3$
$f = 1$

Fig 6 The effect of λ on μ_n, narrow band loading



$\lambda = 1.5E-6, \; n = 3, \; f = 1$

Fig 7 The distribution of crack length under broad band loading "——" result from Eq 24, "o" Monte-Carlo simulation result



λ = 1.5E-6, n = 3, f = 1

Fig 8 The distribution of crack length under broad band loading
"——" result from Eq 30, " o " Monte-Carlo simulation result

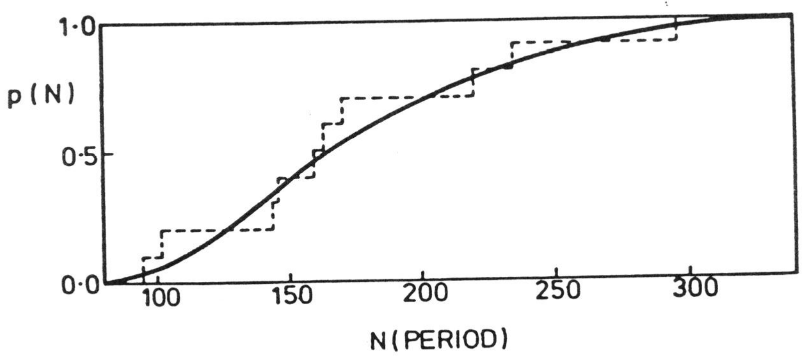


Fig 9 The life distribution
"——" result from Eq 43, "---" Taireja's experiment