INTERACTION OF BRITTLE FRACTURE AND STABILITY LOSS IN BENDED AND COMPRESSED BARS

Zbigniew Kowal*

The interaction curves are determined in dimensionless coordinates, thanks to their special selection and the introduction of parametric function of axial load. The paper is illustrated with examples of integrated interaction curves of brittle fracture and stability loss in bars under axial and transverse load. Effect of geometrical imperfections, load distributed along the bar and concentrated load are considered.

INTRODUCTION

An attempt at solving the problem of estimating the brittle fracture of an eccentrically loaded bar made from elastic-brittle material with different compressive and tensile strengths is reported in Kowal and Laban (1).

From the point of view of the technique of calculation and dimensioning of eccentrically as well as axially and transversely loaded bars, it is essential to determine the boundary curves of brittle fracture and stability loss of such bars.

The present paper reports an attempt at constructing mathematically precise integrated interaction curves of brittle fracture and stability loss of bended and compressed bars in dimensionless coordinates $(S/S_{cr}, M/M_p)$. One integrated curve substitutes the infinite families of interaction curves depending on bar slenderness and the ratio m of brittle fracture strength R to critical stress σ_{cr}:

$$m = \frac{R}{\sigma_{cr}} \qquad (1)$$

* Department of Civil Engineering, TU of Kielce, Poland

The problem was solved owing to the introduction of the parametric moment of brittle fracture M_p as a parametric function of axial load S instead of the moment of brittle fracture that characterizes the bending capacity of the cross section of a brittle bar in the case of zero axial force (S = 0):

$$M_p = \left(R + \frac{S}{F}\right) \frac{J}{C_r} \tag{2}$$

and relation R : σ_{cr} in place of relation R : R_e. By introducing restriction (2) to the condition of the equilibrium of moments we obtained integrated boundary curves. These curves considerably improve the dimensioning technique of brittle bended and compressed bars.

The paper is illustrated with examples of the boundary curves of compressed and transversely loaded bars. Loading conditions are: load equally distributed along the bar, concentrated load, moments concentrated at the ends of the bar and an example of bars with initial curvature. The last example is of essential significance for the dimensioning of bars with geometrical imperfctions.

INTERACTIONS OF BRITTLE FRACTURE AND STABILITY LOSS IN BARS UNDER AXIAL AND TRANSVERSE LOAD

The boundary curve of a simply supported bar. Let's consider a bar under axial and transverse load distributed along a bar as shown in figure 1. The differential equation of the displacement of such a bar has the form:

$$EJ\, y'''' + S\, y'' = q \tag{3}$$

From the solution of the differential equation (3), we obtain the displacement y. In the case of simply-supported ends, we have:

$$y = \frac{B \cos (kx)}{k^4 \cos (k\ell/2)} + \frac{B}{k^2} \left(\frac{x^2}{2} - \frac{1}{k^2} - \frac{\ell^2}{8}\right) \tag{4}$$

where: $B = q/EJ$ and $k^2 = S/EJ$. Brittle fracture will occur at the place of maximal bending moment. Maximal bending moment cannot exceed the brittle fracture strength:

$$\max M = \frac{q\ell^2}{8} \cdot \frac{2(1 - \cos u)}{u^2 \cos u} \leqq M_p = \frac{J}{C_r} \left(R + \frac{S}{F}\right) \tag{5}$$

where $u = k\ell/2 = \sqrt{S/S_{cr}}\, \pi/2$, $S_{cr} = \pi^2 EJ/\ell^2$.

The equation of the interaction of the boundary curve assumes the form:

$$M_p = \frac{q\ell^2}{8} \frac{2(1 - \cos u)}{u^2 \cos u} \tag{6}$$

It is convenient to denote the interaction curve in dimensionless coordinates $[M/M_p, S/S_{cr}]$:

$$\frac{M}{M_p} = \frac{S}{S_{cr}} \cdot \left[\frac{\pi^2}{8}\right] \cdot \frac{\cos\left(\sqrt{S/S_{cr}}\ \pi/2\right)}{1-\cos\left(\sqrt{S/S_{cr}}\ \pi/2\right)} \tag{7}$$

where: $M = ql^2/8$. Restriction of equation (7) is not to exceed the yield point R_e of the material in the compressed area of the cross section of the bar:

$$\sigma = \frac{S}{F} + \frac{M}{J_x} C_s \leq R_e \tag{8}$$

The plot of the interaction curve in coordinates $[M/M_p, S/S_{cr}]$ is represented in figure 1. This plot does not diverge from the straight line. The precise coordinates of curve (7) are given in table 1 (column 2).

The interaction curve of brittle fracture and stability loss can also be introduced in dimensionless coordinates $(q/q_p, S/S_{cr})$. We then obtain:

$$\frac{q}{q_p} = \frac{S}{S_{cr}} \left[\frac{\pi^2}{8}\right] \frac{\cos\left(\sqrt{S/S_{cr}}\ \pi/2\right)}{1 - \cos\left(\sqrt{S/S_{cr}}\ \pi/2\right)} \tag{9}$$

where: $q = 8\ M/\ell^2$ and $q_p = 8\ M_p/\ell^2$

EXAMPLE 1. Calculate the bending strength from the condition of brittle fracture in a bar of length ℓ = 600 cm loaded with axial force $S = 0.6\ S_{cr}$, made from elastic-brittle material with $R = 0.4\ \sigma_{cr}$, the cross section being $F = 18 \times 18 = 324\ cm^2$ and Young's modulus E = 30.000 MPa.

CALCULATION. Critical capacity from the condition of stability loss is:

$$S_{cr} = \pi^2 EJ/\ell^2 = 719.5 \text{ N}$$

Load by axial force is $S = 0.6\, S_{cr} = 431.7$ N.

Critical compressive stress is $\sigma_{cr} = S_{cr}/F = 22.2$ kN/cm².

Critical moment of brittle fracture is:

$$M_p = (R + S/F)\, J/C_r = 9926.46 \text{ kNcm}$$

Critical external load of the bar by moment M is:

$$M = 0.39285\, M_p = 3901 \text{ kNcm}$$

Critical transverse external load of the bar is:

$$q = 8M/\ell^2 = 86.7 \text{ N/cm}$$

CONCLUSION. Critical load of the bar from the condition of brittle fracture is measured by means of coordinates S = 431.7 kN, M = 3901 kNcm or S = 431.7 kN, q = 86.7 N/cm.

In the case of a bar mounted at both ends and under longitudinal and transverse load as shown in figure 2, the maximal bending moment occurs on the support and is:

$$\max M_A = \frac{q\ell^2}{4}\,\frac{\text{tg } u - u}{u^2 \text{ tg } u} \leq M_p \qquad (10)$$

The maximal moment must be less or equal to the capacity of brittle fracture of the cross section. Taking into account the fact that the critical axial load S_{cr} of a bar mounted at both ends differs from that of a simply-supported bar and equals:

$$S_{cr} = 4\pi^2 EJ/\ell^2 \qquad (11)$$

therefore u equals $u = k\ell/2 = \pi\sqrt{S/S_{cr}}$. Finally, the boundary curve assumes the form:

$$\frac{M_A}{M_p} = \frac{\pi^2}{3}\left[\frac{S}{S_{cr}}\right]\frac{\text{tg}\left(\pi\sqrt{S/S_{cr}}\right)}{\text{tg}\left(\pi\sqrt{S/S_{cr}}\right) - \pi\sqrt{S/S_{cr}}} \qquad (12)$$

The coordinates of the interaction curve of brittle fracture and stability loss are given in table 1, (column 4) and the shape is shown in figure 2.

The boundary curve for a span assumes the form:

$$\frac{M_o}{M_p} = \frac{\pi^2}{6}\left[\frac{S}{S_{cr}}\right] \frac{\sin\left(\pi\sqrt{S/S_{cr}}\right)}{\sin\left(\pi\sqrt{S/S_{cr}}\right) - \pi\sqrt{S/S_{cr}}} \tag{13}$$

where: $M_o = ql^2/24$ and $M_A = ql^2/12$.

The coordinates of the boundary curve are represented in table 1 (column 3), the shape is shown in figure 2.

<u>EXAMPLE</u>. Calculate load q of brittle fracture for a support and span assuming that $S = 0.5\ S_{cr}$.

From table 1, we have $M_A = 0.61127\ M_p$ and $M_o = -0.459\ M_p$,

$q = 0.61127 \times 12\ M_p/\ell^2 = 7.335\ M_p\ell^2$.

Load $q = 7.335\ M_p/\ell^2$ resulting from the boundary curve for the cross section on the support is assumed to be adequate for the bar's capacity.

INTERACTION OF BRITTLE FRACTURE AND STABILITY LOSS IN THE CASE OF SINUSOIDALLY DISTRIBUTED TRANSVERSE LOAD

Let's consider a bar loaded axially and transversely by a sinusoidally distributes load. From the solution of the differential equation:

$$EJ\ y'''' + S\ y'' = q_o \sin(\pi x/\ell) \tag{14}$$

the maximal bending moment equals:

$$\max M = \frac{M_o}{1 - S/S_{cr}} \tag{15}$$

The equation of the boundary curve of brittle fracture and stability loss assumes the shape of a straight line:

$$\frac{M}{M_p} + \frac{S}{S_{cr}} = 1 \tag{16}$$

The maximal bending displacement can be determined from the formula:

$$y = y_o / \left(1 - S/S_{cr}\right) \tag{17}$$

where: $y_o = q\ell^4/\pi^4\, EJ$ = deflection of a beam under a transverse load only.

The coordinates of the interaction curve are represented in table 1 (column 5).

<u>THE INTERACTION OF BRITTLE FRACTURE AND STABILITY LOSS OF BARS LOADED AXIALLY AND TRANSVERSELY BY A CONCENTRATED AXIAL FORCE</u>

Let's consider a bar under a load as shown in figure 3. We determine the maximal bending moment of a bar from the differential equation of stability and boundary conditions. This yields:

$$\max M = \frac{P\ell}{4} \cdot \frac{tg\ u}{u} \tag{18}$$

The interaction equation assumes the form:

$$\frac{M}{M_p} = \frac{\sqrt{S/S_{cr}}\ \pi/2}{tg\left(\sqrt{S/S_{cr}}\ \pi/2\right)} \tag{19}$$

The plot of the interaction curve in coordinates $\left(M/M_p,\ S/S_{cr}\right)$ is represented in figure 3. Coordinates of the interaction curves are given in table 2 (column 6).

The equation of the interaction curve in coordinates $\left(P/P_p,\ S/S_{cr}\right)$ has an identical form:

$$\frac{P}{P_p} = \frac{M}{M_p} \tag{20}$$

where: $P = 4\ M/\ell$ and $P_p = 4\ M_p/\ell$.

The displacement in the middle of the bar is determined from the formula:

$$y = \frac{P\ell}{4\ S\ u}\ (tg\ u - u) \tag{21}$$

INTERACTION OF BRITTLE FRACTURE AND STABILITY LOSS OF BARS WITH INITIAL CURVATURE

Let's consider bars eccentrically compressed, simply supported at the ends with an initial sinusoidal curvature $f \sin(\pi x/l)$. In such a case, from the solution of the differential equation of stability loss $EJ y'' + S[y + f \sin(\pi x/\ell)] = 0$ we have:

$$M = S\,y = \frac{S_{cr}}{S_{cr} - S} \cdot S f \sin(\pi x/\ell) \qquad (22)$$

Brittle fracture may occur at the place of the maximal bending moment. Therefore, the boundary equation curve assumes the form:

$$\frac{M}{M_p} + \frac{S}{S_{cr}} = 1 \qquad (23)$$

where: $M = S f$. This is the equation of a straight line. The plot of the interaction curve is the same as that for the sinusoidal transverse load (16). It should be emphasized once more that such a simple equation of the boundary curve of the brittle fracture and stability loss interaction, which replaces the infinite families of the interaction curves, was obtained by the introduction of a parametric function (2) of the cross section's capacity from the condition of brittle fracture and the determination of brittle strength in relation to critical stresses.

EXAMPLE 1. Calculate the moment of brittle fracture of a bar loaded by an axial force $S = 0.6\ S_{cr}$, simply supported at ends, made from elastic-brittle material. The bar is 600 cm long and its cross section is $F = 18 \times 18 = 364\ cm^2$, $J = 8748\ cm^4$, Young's modulus 30,000 MPa and the brittle fracture strength is $R = 0.4\ \sigma_{cr}$.

CALCULATION. Critical capacity from the condition of stability loss equals $S_{cr} = \pi^2\ EJ/\ell^2 = 719$ kN.

The axial load equals $S = 0.6\ S_{cr} = 431.7$ kN.

Critical stress is $\sigma_{cr} = S_{cr}/F = 22.2\ kN/cm^2$.

The brittle fracture strength of the material from the investigation is $R = 8{,}88\ kN/cm^2 = 0.4\ \sigma_{cr}$.

The moment of brittle fracture, when $S = 0$, is:

$$M_p = (R + S/F)\ J/C_r = 9926\ kNcm$$

The moment of brittle fracture, when:

$$S = 0.6\ S_{cr} \text{ is } M = 0.4\ M_p = 0.4 \times 9926 = 3971 \text{ kNcm}$$

The critical value of the initial deflection is:

$$f = M/S = 3971/431.7 = 9.2 \text{ cm}$$

EXAMPLE 2. A bar is given as in Example 1. Calculate the coordinates of the brittle fracture and stability loss interaction in the case of zero compressive strength of the material R = 0. The critical moment of brittle fracture is:

$$M_p = \frac{J}{C_r}\frac{S}{F} = \frac{8748}{9}\frac{431.7}{324} = 1295 \text{ kNcm}$$

The critical load by an external bending moment $M = S\ f$ is:

$$M = 0.4\ M_p = 518 \text{ kNcm}$$

The critical value of the deflection is $f = \frac{518}{431.7} = 1.2$ cm.

INTERACTION OF BRITTLE FRACTURE AND STABILITY LOSS IN A BAR LOADED AXIALLY AND BY CONCENTRATED MOMENTS AT ENDS

Let's consider a bar under a load as shown in figure 4. The maximal bending moment of a bar loaded axially and by moments at the ends has the form:

$$\max M = M_o/\cos u \qquad (24)$$

It is accompanied by a displacement:

$$\max y = (1/\cos u - 1)\ M_o/S \qquad (25)$$

The boundary curve assumes the form:

$$\frac{M_o}{M_p} = \cos\left(\frac{\pi}{2}\sqrt{\frac{S}{S_{cr}}}\right) \qquad (26)$$

The plot of the interaction curve in coordinates $(M_o/M_p,\ S/S_{cr})$ is shown in figure 4. Numerical values are represented in table 1 (column 7).

DISCUSSION AND CONCLUSIONS

1. The introduction of the parametric moment M_p of brittle fracture as a function of axial load made it possible to replace the infinite families of the interaction curves which depend on the shape of the load and the boundary conditions of the bar.

2. The solutions have been illustrated with examples of the interaction curves and the ways of utilizing them in engineering practice.

3. In the construction of the interaction curves it should be remembered that the critical load can be expressed by different formulae for different boundary conditions, which may be easily overlooked due to the formalism of calculation.

4. The interaction curves of stability loss and brittle fracture considerably facilitate the dimensioning of bar pillars built from elastic-brittle materials of tensile strength R smaller than the compressive strength R_e.

SYMBOLS USED

C_r, C_s	distance of the extreme tensile and compressive stresses from the gravity centre of the cross section
E	Young's modulus
F	area of the cross section of the bar
f	arrow of the initial deflection of the bar
J	inertia moment of the cross section
M	moment load on the bar
M_p	moment of brittle fracture
S	axial force in a bar
S_{cr}	critical load of the bar
y	transverse displacement of the bar
σ_{cr}	critical stress of the bar

REFERENCES

1. Kowal, Z., Laban, W., Random strength parameters for elastic plastic brittle compressed bars. Arch. Inz. Lad. T.XXVIII, 23-4/1982.

2. Timoshenko, S.P., Gere, J.M., Theory of elastic stability. Arkady, Warszawa 1983.

TABLE 1 - Coordinates of the interaction curves of brittle fracture and stability loss.

S/S_{cr}	M/M_p	M/M_p	M_A/M_p	M/M_p	M/M_p	M/M_p
1	2	3	4	5	6	7
0.0	1.0	1.0	1.0	1.0	1.0	1.0
0.1	0.89745	-0.88615	0.93226	0.9	0.91637	0.87915
0.2	0.79542	-0.77499	0.86026	0.8	0.82983	0.76324
0.3	0.69386	-0.66665	0.78335	0.7	0.74016	0.65216
0.4	0.59299	-0.56127	0.70069	0.6	0.64711	0.54579
0.5	0.49263	-0.45901	0.61127	0.5	0.55041	0.44402
0.6	0.39285	-0.36002	0.51378	0.4	0.44975	0.34671
0.7	0.29368	-0.26445	0.40655	0.3	0.34479	0.25377
0.8	0.19513	-0.16182	0.28735	0.2	0.23515	0.16507
0.9	0.09723	-0.08427	0.15320	0.1	0.12038	0.08052
1.0	0.0	0.0	0.0	0.0	0.0	0.0

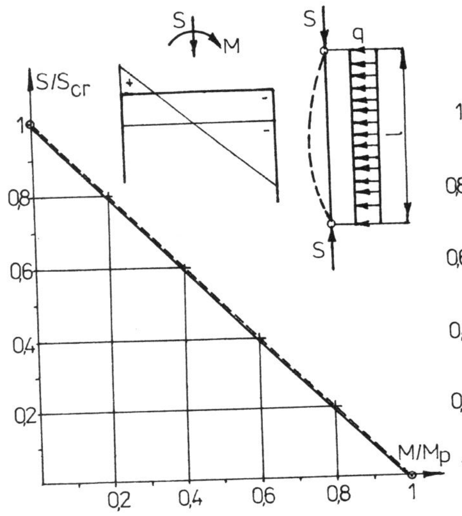

figure 1 Interaction curve (7)

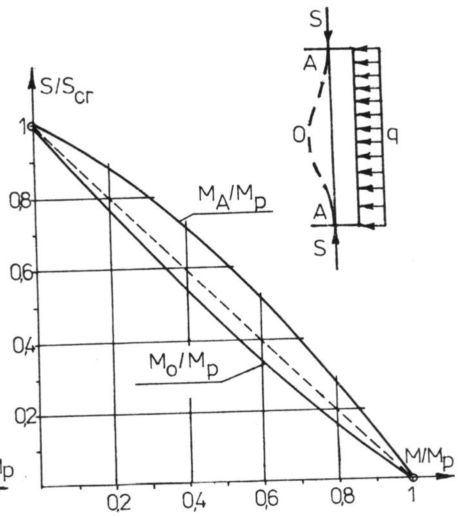

figure 2 Interaction curves (12), (13)

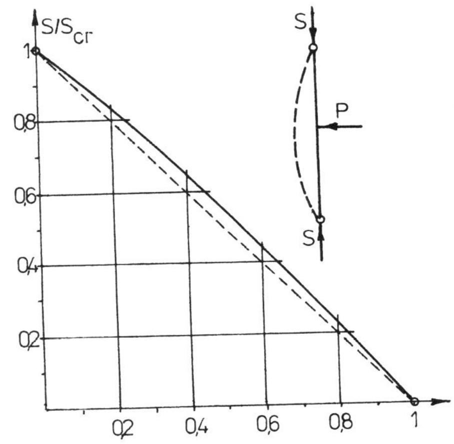

figure 3 Interaction curve (19)

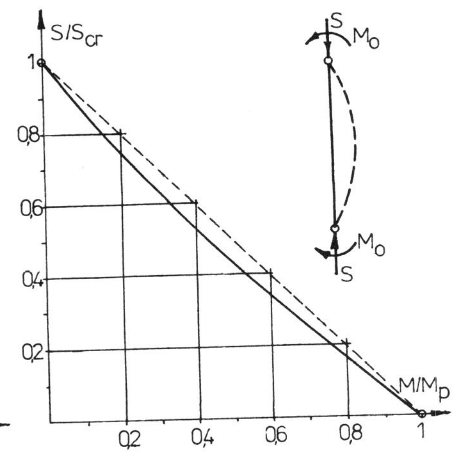

figure 4 Interaction curve (26)