THE OPENING DISPLACEMENT OF A CRACK IN AN INFINITE PLATE SUBJECTED TO CRACK PARALLEL-INITIAL STRESS

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In this paper the COD of a crack in an infinite plate subjected to crack-parallel initial stress is studied. It has been found out that initial crack-parallel tension (compression) has the effect of decreasing (increasing) the COD.

It is wellknown in the classical theory that the crack opening displacement(COD) is independent of a crack-parallel initial stress. Physical intuitions, however, tend to suggest that a crack-parallel tension (compression) should have the effect of decreasing(increasing) the COD of a crack. In this paper the COD of a crack in an infinite plate subjected to crack-parallel initial stress is studied. The theory is based on the Dugdale Barenblatt hypothesis and theory of finite deformation. The plane stress problem is studied.

Let $y_i x_i$ be the coordinates referred to cartesian axes of a point of the reference(initial) configuration B_0 and current configuration B_t respectively. The axis $Ox_3(Oy_3)$ is perpendicular to the midplane of the plate and the crack is on the axis $Ox_1(Oy_1)$. From (3) it is known that:

$$F_{ij} = \frac{\partial y_i}{\partial x_j} \; ; \; C_{ij} = \frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} \tag{1}$$

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Let σ_{ij} be the Piola stress(in(3) it is called the first Piola stress) and W be the strain energy density function for the body in B_o:

$$\sigma_{ij} = 2\frac{\partial y_i}{\partial x_k}\frac{\partial W}{\partial C_{kj}} \tag{2}$$

$$\partial\sigma_{ij}/\partial x_j = 0 \tag{3}$$

Let $\lambda_1\lambda_2\lambda_3$ be the initial stretch ratio of the plate in the direction $0x_1 0x_2 0x_3$ due to the crack-parallel initial stress. In the initial rate:

$$C_{11}=\lambda_1^2;\ C_{22}=\lambda_2^2;\ C_{33}=\lambda_3^2;\ C_{ij}=0\ i\neq j;$$
$$\sigma_{ij_o}=0 \text{ except } \sigma_{11o} \tag{4}$$

where $\sigma_{ij_o}=(\sigma_{ij})_o$ and $(\ )_o$ is $(\ )$ under initial homogeneous deformation.

A small displacement u_i (caused by $\sigma_{22}=\sigma_\infty\lambda_1\lambda_3$ acting at ∞)is superposed on the initial homogeneous deformation:

$$y_i = \lambda_i x_{i\not\Sigma} + u_i;\ u_{i,j} = \frac{\partial u_i}{\partial x_j} < \varepsilon \ll 1 \tag{5}$$

$$C_{ij} = (C_{ij})_o + \delta C_{ij};\ \delta C_{ij} = \lambda_i \lambda_{i,j} + \lambda_j \tag{6}$$

$$\sigma_{ij} = \sigma_{ij_o} + \delta\sigma_{ij};\ \delta\sigma_{ij} = 2W_{jk}u_{i\cdot k} + 2\lambda_i\ W_{ijkl}\delta C_{kl} \tag{7}$$

$$W_{ij} = \left(\frac{\partial W}{\partial C_{ij}}\right)_o;\ W_{ijkl} = \left(\frac{\partial^2 W}{\partial C_{ij}\partial C_{kl}}\right)_o;\ \not\Sigma \text{ no sum} \tag{8}$$

As it will be pointed out in the appendix $\delta\sigma_{33}$; $\delta C_{3\alpha}$ (α=1,2) can be neglected and in the equilibrium equation $\partial\sigma_{\alpha 3}/\partial x_j$ can also be neglected.

$$\delta\sigma_{33} = 0;\ \partial\sigma_{\alpha 3}/\partial x_j = 0;\ \delta C_{3\alpha} = \delta C_{\alpha 3} = 0 \tag{9}$$

Condition $\delta\sigma_{33} = 0$ leads to:

$$2\lambda_3 W_{33\alpha\beta}\delta C_{\alpha\beta} + 2\lambda_3 W_{3333}\,\delta C_{33} = 0 \tag{10}$$

$$\delta C_{33} = -(W_{33\alpha\beta}/W_{3333})\delta C_{\alpha\beta} \qquad \alpha,\beta = 1,2 \tag{11}$$

Substituting (11) into (7) one obtains:

$$\delta\sigma_{ij} = 2W_{jk}u_{i\cdot k} + 2\omega_{ij\alpha\beta}\,\lambda_\alpha\, u_{\alpha\cdot\beta} \tag{12}$$

$$\omega_{ij\alpha\beta} = 2\lambda_{i\not\Sigma} W_{ij} - 2\lambda_3 W_{ij33}W_{33\alpha\beta}/W_{3333} \tag{13}$$

Substituting (12) into (3) one obtains:

$$2W_{jk}u_{i\cdot jk}+2\omega_{ij\alpha\beta}\lambda_\alpha u_{\alpha\cdot\beta j}=0 \qquad i=1,2 \tag{14}$$

Let $\lambda_1 u_1=\omega_{1j2\beta}F_{\cdot\beta j};\ \lambda_2 u_2=-[W_{11}F_{.11}/\lambda_1+\omega_{1j1\beta}F_{.j\beta}]$ (15)

where $F_{,\beta j}=\dfrac{\partial^2 F}{\partial x_\beta \partial x_j}$

Substituting (15) into (14) one obtains:

$$\omega_{2ris}\omega_{ij2\beta}f_{.j\beta rs}-\left(\frac{W_{11}}{\lambda_1}\frac{\partial^2}{\partial x_1^2}+\omega_{1j1\beta}\frac{\partial^2}{\partial x_j\partial x_\beta}\right)\cdot$$

$$\cdot\left(\frac{W_{11}}{\lambda_2}\frac{\partial^2 F}{\partial x_1^2}+\omega_{2r2s}\frac{\partial^2 F}{\partial x_r\partial x_s}\right)=0 \qquad r,s,j,\beta=1,2 \tag{16}$$

Let $F(x_1,x_2)=F(x_1+px_2)$ and substituting it into (16) one obtains the characteristic equation:

$$(\omega_{1j1\beta}\omega_{2r2s}-\omega_{2r1s}\omega_{1j2\beta})p^{r+s+j+\beta-4}+W_{11}^2/(\lambda_1\lambda_2)+$$

$$+(W_{11}\omega_{1j1\beta}/\lambda_2+W_{11}\omega_{2j2\beta}/\lambda_1)p^{j+\beta-2}=0 \tag{17}$$

Equation (17) is an algebraic equation of fourth order. In general, the roots of equation (17) are complex (when there exist real roots, the ellipticity of the equation is lost and unstability occurs. This question will be discussed in another paper). Let $p_1;\ p_2;\ p_3;\ p_4$ be the four roots of equation (17):

$$\begin{matrix}p_1\\p_2\end{matrix}=\alpha_1\pm i\beta_1;\quad \begin{matrix}p_3\\p_4\end{matrix}=\alpha_2\pm i\beta_2;\quad \beta_1>0\quad \beta_2>0 \tag{18}$$

$$F=2\mathrm{Re}\sum_{l=1}^{2}\varphi_i(x_1+p_l x_2);\ u_\alpha=2\mathrm{Re}\sum_{l=1}^{2}a_{\alpha l}\varphi_l(x_1+p_l x_2)$$

$$\delta\sigma_{ij}=2\mathrm{Re}\sum_{l=1}^{2}b_{ijl}\phi_l(x_1+p_l x_2)\ ;\phi_l(x_1+p_l x_2)=$$

$$=\varphi'''_l(x_1+p_l x_2);$$

$$a_{1l}=2\delta_{1j2\beta}p_l^{\beta+J-2}/\lambda_1;\ a_{2l}=-(W_{11}/\lambda_1+\omega_{1j1\beta}p_l^{\beta+j-2})/\lambda_2$$

$$b_{ijl}=2\delta_{1j}W_{11}a_{il}+2\omega_{ij\alpha\beta}\lambda_\alpha a_{\alpha l}p_l^{\beta-1}$$

Now let us study the COD of the crack. At first, the Barenblatt's cohesive stress must be determined. For the convenience of study only the Tresca criterion is considered. In the plastic zone the three true principle stresses are:

$$\sigma_{11}/\lambda_2\lambda_3;\ \sigma_{22}/\lambda_1\lambda_3 = \sigma_o > 0;\ \sigma_{33}/\lambda_1\lambda_2 = 0$$

Thus $\sigma_o = 2k$, $\sigma_{11} > 0$; $\sigma_o = 2k + \sigma_{11}$, $\sigma_{11} < 0$.

where k is the yield shear stress.

Then, let us study the case in which the value of $\delta\sigma_{22}$ on the upper and lower side of the"crack" is equal to $q[\delta(x_1+x_o)+\delta(x_1-x_o)]$, where $\delta(x)$ is the Dirac function. The value of the stress $\delta\sigma_{ij}$ at infinite is zero. On the axis $0x_1\ \delta\sigma_{12}=0;\ b_{121}\Phi_1(x_1)+b_{122}\Phi_2(x_1)=0$. Thus, one obtains(1) (Liebowitz. H.):

$$\Phi_1(x_1) = -b_{122}\Phi_2(x_1)/b_{121}$$

$$\delta\sigma_{22} = 2\,\mathrm{Re}\sum_{l=1}^{2} b_{22l}\Phi_l(x_1) = B\Phi_2(x_1) + \bar{B}\overline{\Phi_2(x_1)} = \tag{19}$$

$$= q[\delta(x+x_o)+\delta(x_1-x_o)];\ |x_1|<c=a+R;\ B=b_{222}-b_{221}\frac{b_{122}}{b_{121}}$$

where 2c is the length of the "crack"; 2a is the length of the true crack; R is the plastic zone size; R=c-a.

$$[B\Phi_2(x_1)-\bar{B}\bar{\Phi}_2(x_1)]^+ - [B\Phi_2(x_1)-\bar{B}\bar{\Phi}_2(x_1)]^- = 0 \qquad |x_1|<c \tag{20}$$

$$[B\Phi_2(x_1)+\bar{B}\bar{\Phi}_2(x_1)]^+ + [B\Phi_2(x_1)+\bar{B}\bar{\Phi}_2(x_1)]^- =$$

$$= 2q\,[\delta(x_1+x_o)+\delta(x_1-x_o)] \qquad |x_1|<c \tag{21}$$

$$B\Phi_2(z_2)-\bar{B}\bar{\Phi}_2(z_2) = 0 \qquad z_2 = x_1 + p_2x_2 \tag{22}$$

$$B\Phi_2(z_2) = qz_2 i\sqrt{c^2-x_o^2}/[\pi i\sqrt{z_2^2-c^2}(x_o^2-z_2^2)] \tag{23}$$

$$k_1 = \lim_{x_1\to c}\sqrt{2(x_1-c)}\,\sigma_{22}/\lambda_1\lambda_3 = \lim_{x_1\to c}\sqrt{2(x_1-c)}\lambda_1^{-1}\lambda_3^{-1}\mathrm{Re}[B\Phi_2(x_1)] =$$

$$= 2q\sqrt{c}/[\lambda_1\lambda_3\pi\sqrt{c^2-x_o^2}] \tag{24}$$

$$B\varphi_2''(z_2) = \int\Phi_2(z_2)dz_2 = \frac{q}{2\pi i}[\ln(\sqrt{c^2-z_2^2}+\sqrt{c^2-x_o^2}) -$$

$$-\ln(\sqrt{c^2-z_2^2} - \sqrt{c^2-x_o^2})] + c_o \tag{25}$$

$$\mathrm{COD}=2u_2^+(x_1=a)=4\mathrm{Re}\sum_{i=1}^{2}a_{2i}\varphi_i''(a)=4\mathrm{Re}[(a_{22}-a_{21}\frac{b_{122}}{b_{121}})\varphi_2''(a)]$$

$$=4\mathrm{Re}\frac{\gamma q}{2\pi i}\ln\left(\sqrt{c^2-a^2}+\sqrt{c^2-x_0^2}\right)/\left(\sqrt{c^2-a^2}-\sqrt{c^2-x_2^2}\right) \qquad (26)$$

where $\gamma=(a_{22}-a_{21}\frac{b_{122}}{b_{121}})/B$

Then, let us study the case in which the value of $\delta\sigma_{22}$ on the upper and lower side of the "crack" is equal to $0(|x_1|<a)$ and $\sigma_0\lambda_1\lambda_3(|x_1|>a)$.

$$k_1^1=-(2\sqrt{c}/\pi)\int_a^c(\sigma_0/\sqrt{c^2-x_0^2})dx_0=-2\sqrt{c}\,\sigma_0\arccos(a/c)/\pi$$

$$\mathrm{COD}^1=2\mathrm{Im}(\gamma/\pi)\sigma_0\lambda_1\lambda_3\int_a^c\ln\frac{\sqrt{c^2-a^2}+\sqrt{c^2-x^2}}{\sqrt{c^2-a^2}-\sqrt{c^2-x^2}}dx_0 \qquad (27)$$

$$=4\sigma_0\lambda_1\lambda_3(\sqrt{c^2-a^2}\arccos(a/c)-a\ln(c/a))\mathrm{Im}(\gamma)/\pi$$

At last, let us study the case in which the value of $\delta\sigma_{22}$ on the upper and lower side of the "crack" is equal to $\sigma_\infty\lambda_1\lambda_3$:

$$k_1^2=2\sigma_\infty(\sqrt{c}/\pi)\int_0^c(c^2-x_0^2)^{-0.5}dx_0=\sqrt{c}\,\sigma_\infty/\lambda_1\lambda_3 \qquad (28)$$

$$\mathrm{COD}^2=-(2/\pi)\sigma_\infty\lambda_1\lambda_3\mathrm{Im}(\gamma)\int_0^c\ln\frac{\sqrt{c^2-a^2}+\sqrt{c^2-x_0^2}}{\sqrt{c^2-a^2}-\sqrt{c^2-x_0^2}}dx_0=$$

$$=2\mathrm{Im}(-\gamma)\sigma_\infty\lambda_1\lambda_3\sqrt{c^2-a^2} \qquad (29)$$

$$k_1=k_1^1+k_1^2=0;\quad a/c=\cos(\pi\sigma_\infty/2\sigma_0);\ R=c-a=a\sec(\pi\sigma_\infty/2\sigma_0)-a \qquad (30)$$

$$\mathrm{COD}=\mathrm{COD}^1+\mathrm{COD}^2=4\sigma_0 a\mathrm{Im}(-\gamma)\ln\sec(\pi\sigma_\infty/2\sigma_0)\lambda_1\lambda_3/\pi \qquad (31)$$

For example, let $W=\lambda(C_{kk}-3)^2/8+\mu(C_{ij}-\delta_{ij})(C_{ij}-\delta_{ij})/4$

$\lambda=\mu$. COD= $\beta\,\mathrm{COD}_c$ where COD_c=COD in classical theory and $\sigma=2\sigma_{11o}/\mu\lambda_2\lambda_3$.

σ=-0.1,-0.05,-0.025,0,0.025,0.05,0.10,0.16,0.20
λ_1=0.98, 0.99, 0.995,1,1.005,1.01,1.02,1.03,1.04
β=1.03, 1.02, 1.01 ,1,0.995,0.99,0.98,0.96,0.95

As it is pointed out in the example, the initial crack-parallel tension(compression) has the effect of decreasing(increasing) the COD.

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APPENDIX

Let h be the thickness of the plate and l be a characteristic length dimension of the plate:

$$\eta = h/l \ll 1 \quad \overline{\delta\sigma_{ij}} = \delta\sigma_{ij}/\mu \quad \bar{x}_i = x_i/l \quad \bar{u}_i = u_i/l \quad |\bar{x}_3| \leq \frac{h}{2l} = \frac{\eta}{2}$$

Using Taylor expansion, one has obtained:

$$\bar{u}_i = \sum_{n=0}^{\infty} u_{i.n}\bar{x}_3^n/n!\,;\ \overline{\delta\sigma_{ij}} = \sum_{n=0}^{\infty}\overline{\delta\sigma_{ij.n}}\ \bar{x}_3^n/n! \quad \text{where } f_{.n} = \left(\frac{\partial^n f}{\partial x_3^n}\right)_{x_3=0}$$

Because of the symmetricity:

$$\overline{\delta\sigma}_{33.2n+1} = 0;\ \overline{\delta\sigma}_{3\alpha.2n} = 0;\ \overline{\delta\sigma}_{\alpha3.2n} = 0;\ \bar{u}_{\alpha.2n+1} = 0;$$

$$\frac{\partial\bar{u}_\alpha}{\partial x_3} = 0;\ \bar{u}_{3.2n} = 0.$$

Let $(\bar{\ }) = \frac{1}{\eta}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}(\ )\,d\bar{x}_3 \quad \overline{\overline{\delta\sigma_{ij}}} = 2W_{jk}\bar{\bar{u}}_{i.k} + 2\lambda_i W_{ijlm}\delta\bar{\bar{C}}_{lm};$

$$\because \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\bar{x}_3^{2n+1}dx_3 = 0 \quad \bar{\bar{u}}_{3.\alpha} = 0$$

$$\delta\bar{\sigma}_{33} = \delta\bar{\sigma}_{33.0} + \delta\bar{\sigma}_{33.2}\bar{x}_3^2/2 + 0(\eta^4);\ \bar{x}_3 = \pm\frac{\eta}{2}:\ \delta\overline{\sigma}_{33} = 0$$

$$\delta\bar{\sigma}_{33} = 0(\eta^2)$$

The quantities $0(\eta^2)$ can be neglected.

$$\delta\overline{\sigma}_{33} = 0$$

In main text, $\bar{f}$ and $\bar{\bar{f}}$ are denoted by f for convenience.

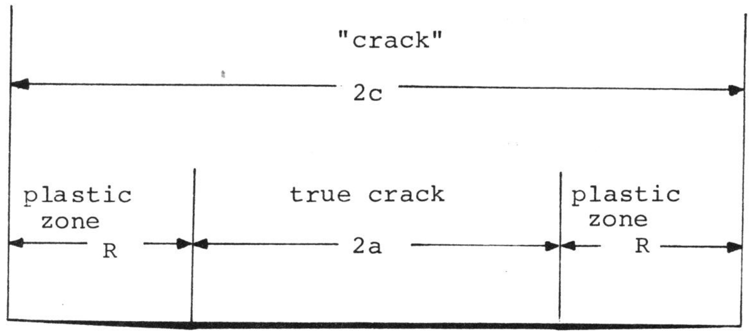

Figure 1 True crack, "crack" and plastic zone