CURVILINEAR CRACKS IN PLANAR SITUATIONS

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This paper provides formulae for the geometrical parameters (branching angle, curvature) of a propagating crack in the most general planar situation. The treatment is based on :
i- a calculation of the stress intensity factors at the tip of a small virtual crack extension of arbitrary shape ;
ii- a discussion of propagation criteria, which concludes that the so-called "principle of local symmetry" is the only possible one for purely logical reasons.
The paper provides also a solution to the problem of the eventual coïncidence of this principle and the Griffith criterion.

1 - INTRODUCTION

The prediction of crack paths in the most general planar situations (arbitrary geometry of the body considered and the of initial crack, arbitrary loading), in the context of LEFM, involves necessarily two aspects:

i - The computation of stress intensity factors and other related parameters for a given body, a given crack and a given loading. This step can only be numerical since no explicit formula will ever give stress intensity factors for arbitrary geometries and loadings.

ii- The definition of the crack path in the immediate future, by means of formulae giving geometrical parameters of the crack extension such as the branching angle (in case of load discontinuity) or the initial curvature (in case of regular propagation, i.e. continuously turning tangent) in terms of the previously computed stress intensity factors and related parameters. This step is necessarily analytical for reasons explained below.

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The prediction of crack paths is then conducted by a step-by-step method.

The aim of the present paper is to provide a complete (i.e. valid in the most general case) solution to the theoretical problem (ii). This necessitates :

(ii.a) The calculation of the stress intensity factors at the tip of a small virtual crack extension of arbitrary shape, in terms of previous stress intensity factors and related parameters and parameters characterizing the local geometry of the initial crack and of its extension. This can only be done analytically and not numerically since the geometry of the crack extension is not known a priori.

(ii.b) The formulation of a propagation criterion expressed in terms of the stress intensity factors at the tip of a virtual extension. Combination of this criterion and the expressions derived in (ii.a) yields then formulae for the geometrical parameters of the crack extension.

Previous calculations of the stress intensity factors at the tip of a small crack extension were based either on exact analytical methods applicable only for particular geometries and loadings (see e.g. Wu [1] or Bilby et al. [2]) or on perturbation techniques valid for nearly straight cracks (see e.g. Cotterell and Rice [3] or Sumi et al. [4]). The objective of this work being to treat the most general case, it is obvious that radically different methods are needed. The present approach consists in establishing first (section 2) the general form of the successive terms of the expansion of the stress intensity factors in powers of the crack extension length, i.e. in making precise on which parameters they depend; this will be done by general considerations based on scale changes. As will be seen, most terms have universal expressions, in the sense that they are given by formulae involving only the parameters characterizing the local geometry of the crack and its extension and the initial stress field near the crack tip, valid whatever the geometry of the body under consideration and the prescribed forces or displacements. The various functions appearing in the expansion are then identified (section 3) by two different techniques: terms describing the expansion of the stress intensity factors for a straight extension are calculated by studying the special case of a straight initial crack in an infinite body submitted to uniform forces at infinity by Muskhelishvili's method; terms describing the effect of the crack extension curvature are obtained by modelling the curved extension as a succession of n straight segments and letting n tend towards infinity.

Propagation criteria are next discussed (section 4). A criterion is said to verify property (P) if it predicts a zero branching angle only when the second stress intensity factor just before the eventual change of direction is zero. It is shown that any criterion verifying property (P) must either be identical to the so-called "principle of local symmetry" (PLS) according to which the second stress intensity factor must be zero immediately after the eventual change of direction, or be logically inconsistent. This leads at once to rejection of some criteria [5,6] wich agree with (P) but differ from the PLS. The case of the more fundamental Griffith criterion (maximum energy release rate along the propagation direction) (1) is next considered; the problem of its eventual coïncidence with the PLS is solved by means of a calculation of the expansion of the functions connecting the stress intensity factors just before and just after a change of direction. It is concluded that the two criteria are definitely different, though extremely close numerically ; since Griffith's criterion agrees with (P), this implies that it is logically inconsistent. The PLS appears thus as the only possible criterion for purely logical reasons. It is finally used to derive formulae for the geometrical parameters of the crack extension.

2 - GENERAL FORM OF THE EXPANSION OF THE STRESS INTENSITY FACTORS

We consider (fig. 1) a two-dimensional linear elastic body of arbitrary shape containing a curvilinear crack and subjected to prescribed forces or displacements on its boundary. The tangent to the crack at its tip 0 is denoted 0x, and the local curvature C. We consider a virtual deviated and curved extension of length s ; the kink angle is denoted πm, and the distance between a point on the crack extension and its projection on the deviated tangent at the point 0 is taken of the form $a^* s^{3/2} + \frac{C^*}{2} s^2 + o(s^2)$ (2). The case of regular propagation will be treated as a particular case where $m = 0$ and $a^* = 0$. We want to derive the expansion of the stress intensity factors at the tip of the crack extension $k(s) = (k_1(s), k_2(s))$ in successive powers of s up to the third term (proportional to s). There would be no fundamental difficulty in extending the present approach to higher order terms, but we do not think it would be very useful since, as will be seen below, the third term is sufficient to obtain the expression of the curvature of the crack at any regular point, and thus to describe not only any kinks that may occur but the entire propagation.

2.1 - First term of the expansion (proportional to $s^0 = 1$) :

The first term consists of the stress intensity factors $k^* = (k_1^*, k_2^*)$ immediately after the kink. It will be shown that they depend only on the stress intensity factors $k = (k_1, k_2)$ just before the kink and the branching angle πm, and not on other parameters such as a^*, C, C^*,... This means that the relation $k^* = F(m,k)$ esta-

(1) This denomination, though widely used, is in fact improper, since this criterion goes very probably beyond Griffith's original ideas. It is nevertheless used here for shortness.

(2) Such a form, which implies an infinite curvature at the point 0, is necessary in order to respect the criterion (PLS) ; see e.g. [3,4] and below.

blished in [1,2,7,10] in a particular case (straight initial crack, straight extension, infinite body, uniform forces at infinity) has in fact a universal value.

We consider first the case where the body is a circular disk of center 0 and radius R, subjected to a force field $T = \{ t \}$, $t = (\sigma_{rr}, \sigma_{r\theta})$ on its boundary (r, θ : polar coordinates with respect to the 0x axis). The stress intensity factors at the tip of the crack extension of length s can be expressed as :

$$k(s) = L(m,R,C,a^*,C^*,s,T) \tag{1}$$

where L is a linear functional of the stress field T, depending on the geometrical parameters m, R, C, a*, C*, s(1).

Let a new structure be homothetic to the first one by a factor λ : m, R, C, a*, C*, s become then m, λR, C/λ, $a^*/\sqrt{\lambda}$, C^*/λ, λs. If it is subjected to the same force field, the stresses are the same (at homothetic points) ; the stress intensity factors are then easily seen to be greater by a factor $\sqrt{\lambda}$ in the new structure : thus

$$L(m,\lambda R,C/\lambda,a^*/\sqrt{\lambda},C^*/\lambda, \lambda s,T) = \sqrt{\lambda}\ L(m,R,C,a^*,C^*,s,T) \tag{2}$$

Let $L^*(m,R,C,a^*,C^*,.)$ be the limit of the functional $L(m,R,C,a^*,C^*,s,.)$ when s is shrunk to zero (this is the functional that gives k*). Letting s tend towards zero in (2), we obtain a similar "homogeneity" property for L^* :

$$L^*(m,\lambda R,C/\lambda,a^*/\sqrt{\lambda},C^*/\lambda,T) = \sqrt{\lambda}\ L^*(m,R,C,a^*,C^*,T) \tag{3}$$

We now come back to the general case (fig. 1). Let $T(R,s)$ be the force field applied on the boundary of the circular disk of center 0 and radius R when the crack extension length is s and when the prescribed forces or displacements are applied on the external boundary. The stress intensity factors k(s) are unaffected if one eliminates the exterior of the disk of radius R while exerting the force field $T(R,s)$ on its boundary. Thus, by equ. (1) :

$$k(s) = L(m,R,C,a^*,C^*,s, T(R,s)) \tag{4}$$

In order to take the limit $s \to 0$ in equ. (4), we introduce the following proposition :

Proposition 1 : at a fixed point, the stresses are continuous when the kink occurs ; this means that when s is shrunk to zero, $T(R,s)$ tends towards the field force $T(R)$ exerted on the boundary of the disk of radius R before the kink(2).

(1) Other geometrical parameters, such as the derivatives of the curvature C, have been omitted for simplicity in the notation; introducing them would lead to the same result.

(2) But of course after the load discontinuity responsible for the kink.

Proof : we consider the structure in two situations : before the kink and when the crack extension length is s. In the first situation the crack is considered as extending up to the same point as in the second one, the crack extension being closed by suitable applied forces. If we take the difference between the two problems, the field force exerted on the boundary of the disk of radius R is $T(R,s) - T(R)$. In the limit $s \to 0$, the boundary conditions of this new problem tend towards zero : indeed the forces or displacements prescribed on the external boundary in the original problem are supposed to vary continuously with s (after the load discontinuity), and the total forces exerted on the lips of the crack extension are $0(\sqrt{s})$ since the forces per unit length are $0(1/\sqrt{r})$ and must be integrated over a distance s. Thus the stresses at any point tend towards zero, which means that $\lim_{s \to 0} [\,T(R,s) - T(R)\,] = 0$, QED.

Taking the definition of L^* plus this proposition into account, equ. (4) becomes for $s \to 0$:

$$k^* = \lim_{s \to 0} k(s) = L^*(m,R,C,a^*,C^*, T(R)) \tag{5}$$

Thus k* depends only on the stress field before the kink ; this is the essential property which will ensure the universality of its expression.

To obtain this expression, we use equ.(3) plus the linearity of L^* with respect to T to rewrite equ.(5) as :

$$k^* = \sqrt{R}\, L^*(m,1,R\,C,\sqrt{R}\,a^*,R\,C^*,T(R)) = L^*(m,1,R\,C,\sqrt{R}\,a^*,\,R\,C^*,\sqrt{R}\,T(R)) \tag{6}$$

We let now R tend towards zero. Since the force per unit length exerted on the boundary of the disk of radius R before the kink is of the form :

$$t(R,\theta) = \frac{k_1\, f_1(\theta)}{\sqrt{R}} + \frac{k_2\, f_2(\theta)}{\sqrt{R}} + 0(1) \quad ,$$

where f_1 and f_2 are universal vectorial functions, $\sqrt{R}\,T(R)$ tends towards the force field $\{\, k_1 f_1(\theta) + k_2 f_2(\theta) \,\}$ when R is shrunk to zero. Thus equ.(6) becomes in the limit $R \to 0$:

$$k^* = L^*\left[\, m,1,0,0,0, \left\{ k_1\, f_1(\theta) + k_2\, f_2(\theta) \right\} \right] = F(m,k) \tag{7}$$

where F is a (vectorial) function depending only on m and k, linear in k, which is the result announced. The final disappearance of all curvature parameters in the expression of k* comes from the fact that the crack and its extension "appear as straight" in the limit of infinitely small disks.

Though the validity of equ. (7) had not been established previously in a fully general case, Cotterell and Rice [3] had proved it for a nearly straight initial crack with a small branching angle in an infinite body, and Sumi et al. [4] for a straight initial crack with a small branching angle in an arbitrary body.

In [10], the form (7) is given for a straight kink on a semi-infinite crack for any m.

2.2 - Second term of the expansion (proportional to $\sqrt{s}$) :

In contrast to the first term of the expansion, the next ones are influenced by the (continuous) variation of the prescribed forces or displacements. We deal however for simplicity with a constant loading ; extension to variable ones is straightforward. In the particular case of a proportional loading, it can be shown that the final expressions for m, a*, C* are exactly the same as for a constant loading (though this is not true for the formulae giving k(s)).

The derivation of the second term is based on the same principle as that of the first one. The functional L is expanded in powers of s :

$$\left.\begin{aligned} &L(m,R,C,a^*,C^*,s,.) = \\ &\quad L^*(m,R,C,.) + L^{(1/2)}(m,R,C,a^*,C^*,.)\sqrt{s} + o(\sqrt{s})^{(1)} \end{aligned}\right\} \quad (8)$$

The $\sqrt{s}$ dependence of the second term of the expansion may seem arbitrary ; in fact, assuming only that L can be expanded in some powers of s (not necessarily s^0, $s^{1/2}$, s^1,...), it can be shown that the expansion contains no term proportional to s^α with $0 < \alpha < 1/2$; thus it must have the form given by equ. (8) ($L^{(1/2)}$ being possibly zero). The proof shows in fact that this form is directly dictated by that of the expansion of the stresses near a crack tip.

Using this equation plus the "homogeneity" property of the functional L (equ. (2)), one obtains a slightly different "homogeneity" property for $L^{(1/2)}$:

$$L^{(1/2)}(m,\lambda R,C/\lambda,a^*/\sqrt{\lambda},C^*/\lambda,T) = L^{(1/2)}(m,R,C,a^*,C^*,T) \quad (9)$$

We now introduce a new proposition which refines Proposition 1, and can be proved using the theory of Bueckner's "weight functions" :

Proposition 2 : at a given point, the stresses are a differentiable function of the crack extension length. Thus the expansion of $T(R,s)$ in powers of s has the form :

$$T(R,s) = T(R) + T^{(1)}(R)s + o(s) \quad (10)$$

without any $\sqrt{s}$-term.

The expansion of k(s) is then easily deduced from equs. (4), (8), (10) and the linearity of L with respect to T :

$$k(s) = k^* + k^{(1/2)}\sqrt{s} + o(\sqrt{s}) \quad , \quad \text{where :}$$

$$k^{(1/2)} = L^{(1/2)}(m,R,C,a^*,C^*,T(R)) \quad (11)$$

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(1) The parameters a* and C* are now omitted in L^* ; indeed, since k* depends only on m and k, which depends on R, C and T but not on a* and C*, L^* is independent of a* and C*.

Here again it is seen that $k^{(1/2)}$ depends only on the stress field before the kink, and this will ensure the universality of its expression. This property is a direct consequence of the absence of a $\sqrt{s}$-term in the expansion of $T(R,s)$.

Using equ. (9), we rewrite equ. (11) as

$$k^{(1/2)} = L^{(1/2)}(m,1,R\,C,\sqrt{R}\,a^*,R\,C^*,\,T(R)) \qquad (12)$$

We will now expand equ. (12) in powers of R. We need for this the expansion of $T(R)$, i.e. that of $t(R,\theta)$:

$$t(R,\theta) = \frac{k_1\, f_1(\theta)}{\sqrt{R}} + \frac{k_2\, f_2(\theta)}{\sqrt{R}} + T\, g(\theta) + O(\sqrt{R}) \quad ,$$

where T is the non-singular stress before the kink and g a universal (vectorial) function. Using this equation and expanding the functional $L^{(1/2)}(m,1,R\,C,\sqrt{R}\,a^*,R\,C^*,\cdot)$ in powers of R, one puts equ. (12) under the form :

$$\left.\begin{aligned} k^{(1/2)} &= L^{(1/2)}\left[m,1,0,0,0,\left\{k_1 f_1(\theta)+k_2 f_2(\theta)\right\}\right]\frac{1}{\sqrt{R}} \\ &\quad + L^{(1/2)}\left[m,1,0,0,0,\left\{T\, g(\theta)\right\}\right] \\ &\quad + a^*\frac{\partial L^{(1/2)}}{\partial a^*}\left[m,1,0,0,0,\left\{k_1\, f_1(\theta)+k_2\, f_2(\theta)\right\}\right] + O(\sqrt{R}) \end{aligned}\right\} .$$

This equation holds for every R, which means that the right-hand side is in fact independent of R. Therefore the divergent $1/\sqrt{R}$-term must be zero. In the limit $R\to 0$, one obtains thus :

$$\begin{aligned} k^{(1/2)} &= L^{(1/2)}\left[m,1,0,0,0,\left\{T\, g(\theta)\right\}\right] \\ &\quad + a^*\frac{\partial L^{(1/2)}}{\partial a^*}\left[m,1,0,0,0,\left\{k_1\, f_1(\theta)+k_2\, f_2(\theta)\right\}\right] = T\, G(m) + a^*\, H(m,k) \end{aligned} \qquad (13)$$

where G depends only on m and H only on m and (linearly) on k.

Hence $k^{(1/2)}$ has a universal expression like k^* (though more complex). An expression of this type was found for instance by Sumi et al. [4] in the particular case of a straight initial crack with small parameters m, a^*, C^*. However in this work this expression appeared as the beginning of an expansion in powers of m, a^*, C^* ; equ. (13) shows that remarkably enough, the only powers of a^* and C^* which intervene in $k^{(1/2)}$ are 1 and a^*.

2.3- Third term of the expansion (proportional to s) :

For space reasons, we give here only the result for the third term. The derivation is basically the same as those presented in §2.1 and §2.2, all expansions being now carried to a higher order ; however new complications are encountered : for instance it can be shown

that due to the curvature of the crack, the third term of the expansion of the stresses near the (initial) crack tip (proportional to $\sqrt{R}$) comprises not only a term $[b_1 h_1(\theta) + b_2 h_2(\theta)]\sqrt{R}$ as for a straight crack, but also a term of the form $[k_1 C \tilde{h}_1(\theta)+k_2 C \tilde{h}_2(\theta)]\sqrt{R}$ (C denoting as above the crack curvature at its (initial) tip).
The result is

$$k(s) = k^* + k^{(1/2)}\sqrt{s} + k^{(1)} s + o(s) \tag{14}$$

where :

$$\left.\begin{aligned} k^{(1)} = & \underset{R \to 0}{PP}\ L^*(m,1,R\,C,\sqrt{R}\,T^{(1)}(R)) \\ & + I(m,b) + C\,J(m,k) + a^*\,T\,K(m) + a^{*2}\,L(m,k) + C^*\,M(m,k) \end{aligned}\right\} \tag{15}$$

In this equation, the first term of the right-hand side denotes the principal part of $L^*(m,1,R\,C,\sqrt{R}\,T^{(1)}(R))$ when R tends towards zero, i.e. its limit once its divergent part has been substracted ; $T^{(1)}(R)$ denotes as above the derivative of $T(R,s)$ with respect to s ; b is the vector (b_1, b_2) ; and I, J, K, L, M denote (vectorial) functions which depend only on the indicated parameters (I being linear with respect to b and J, L, M with respect to k). Like equ. (13), equ. (15) is not the beginning of an expansion in powers of a^* and C^* but an exact expression.

The essential difference between the expression of $k^{(1)}$ and those of k^* and $k^{(1/2)}$ is the appearance, in addition to terms depending on the stress field $T(R)$ before the kink, of a term depending on $T^{(1)}(R)$ and thus on the stress field after the kink. It can be shown that this term is non-universal in the sense that its does not depend only on the parameters describing the local geometry of the crack and its extension (m, C, a^*, C^*, ...) but on the whole geometry of the body under consideration (which means that it must be calculated for each particular case). More precisely, a more detailed examination of $T^{(1)}(R)$ based on the theory of Bueckner's weight functions shows that this term has the following form (in matrix notation) :

$$\underset{R\to 0}{PP}\ L^*(m,1,R\,C,\sqrt{R}\,T^{(1)}(R)) = [F(m)]\,[A][F(m)]^T[F(m)]\,k^{(1)} \tag{16}$$

where $[F(m)]$ is the 2x2 matrix defining the F function ($k_i = F_{ij}(m)\,k_j$), $[F(m)]^T$ is its transpose, and [A] is a 2x2 matrix which depends on the whole geometry of the body and the crack before the kink, and on which portions of the boundary the prescribed forces and displacements are applied, but not on the geometrical parameters of the crack extension (m,a^*,C^*) nor on the loading. Thus even in this term the influences of m and the loading appear under a universal form (through the [F(m)] matrix and the stress intensity factors $k = (k_1, k_2)$); the non-universality appears only in the geometry dependent matrix [A].

(1) In this equation k must of course be considered as a column vector.

These results are a confirmation and an extension of those of Sumi et al., who were the first to note [4] the loss of the universality property in $k^{(1)}$ (in the particular case of a straight initial crack and small parameters m, a*, C*), and obtained [8] an expression of the non-universal term accurate to the first order in m which coïncides with that deduced from (16) (once the F function is known : see § 3.1 below).

We will now derive another, more interesting, form of equ. (15). For this we remark that for a straight ($a^* = 0$, $C^* = 0$) extension in the direction πm, $k^{(1)}$ reduces to :

$$[k^{(1)}]_{\substack{\text{straight}\\ \pi m}} = \underset{R\to 0}{P\,P}\; L^*(m,1,R\,C,\sqrt{R}\;T^{(1)}(R)) + I\,(m,b) + C\,J\,(m,k)$$

If we introduce now non-zero curvature parameters a*, C*, this does not change the first three terms of equ. (15) (for the first one this results from equ. (16)) but just introduces new terms, so that

$$k^{(1)} = [k^{(1)}]_{\substack{\text{straight}\\ \pi m}} + a^*\,T\,K(m) + a^{*2}\,L(m,k) + C^*\,M\,(m,k) \qquad (17).$$

Equation (17) exhibits a decomposition of $k^{(1)}$ into a term which is non-universal but independent of a* and C* plus universal terms which depend on these parameters. Despite the non-universality of $[k^{(1)}]_{\substack{\text{straight}\\ \pi m}}$, this equation is interesting in order to derive the value of C* (using the propagation criterion) because $[k^{(1)}]_{\substack{\text{straight}\\ \pi m}}$ can be computed numerically in each particular case, the essential point being that it is independent of C* which is unknown a priori. The form (17) is more convenient for practical purposes than (15) because $[k^{(1)}]_{\substack{\text{straight}\\ \pi m}}$ is easily evaluated by computing the stress intensity factors at the tip of a small straight extension in the direction πm, whereas the numerical evaluation of a parameter such as b is difficult.

3 - IDENTIFICATION OF THE FUNCTIONS F, G, H, K, L, M

3.1 - Identification of the functions F and G :

These functions describe the first two terms of the expansion of k(s) in the case of a straight extension ($a^* = 0$, $C^* = 0$) (see equs. (7) and (13)). Owing to their universality property, one can evaluate them by considering the special case of a straight initial crack of length 2ℓ placed in an infinite body subjected to uniform forces at infinity (fig. 2). Following Muskhelishvili's method, we introduce the conformal transformation $Z = \omega(z)$ which maps the exterior of the unit circle (variable z) onto the physical domain (variable Z) (fig. 3) ; this transformation is defined by :

$$\omega(z) = Re^{im\alpha} (z - e^{i\alpha})^{1-m} (z - e^{-i\alpha})^{1+m} / z \tag{18}$$

where R, α and β are defined by :

$$\ell = 2 R \left(\cos \frac{\alpha + \beta}{2} \right)^{1-m} \left(\cos \frac{\alpha - \beta}{2} \right)^{1+m} \tag{19}$$

$$s = 4 R \left(\sin \frac{\alpha + \beta}{2} \right)^{1+m} \left(\sin \frac{\alpha - \beta}{2} \right)^{1-m} \tag{20}$$

$$\sin \beta = m \sin \alpha \tag{21}$$

The stress intensity factors $k_1(s)$, $k_2(s)$ are then given [7] by :

$$k_1(s) - i\, k_2(s) = 2\sqrt{\pi}\, \varphi'(e^{i\beta})\, e^{-i\pi m/2} [\, \omega''(e^{i\beta})\,]^{-1/2} \tag{22}$$

where φ is Muskhelishvili's first complex potential (in the transformed plane), satisfying the integral equation:

$$\varphi(z) = \Gamma\, Re^{im\alpha} z - (\Gamma + \overline{\Gamma}') \frac{Re^{-im\alpha}}{z} + \frac{1 - e^{2i\pi m}}{2i\pi} \int_C \frac{(\lambda - e^{i\alpha})\,(\lambda - e^{-i\alpha})\,\overline{\varphi'(\lambda)}\, d\lambda}{\lambda(\lambda + e^{-i\beta})(\lambda - e^{i\beta})(\lambda - z)} \tag{23}$$

In this equation C is the arc shown on fig. 3, going round the point $e^{i\beta}$ through the interior of the unit circle, and Γ and Γ' are given by :

$$\Gamma = \frac{N_1 + N_2}{4} \quad ; \quad \Gamma' = \frac{(N_2 - N_1)\, e^{-2i\gamma}}{2} \tag{24}$$

where N_1 and N_2 are the principal stresses at infinity and γ the angle between the initial crack and the first principal direction (fig. 2). Equation (23) was first obtained by Hussain et al. [9]. It is easy to show from equs. (19 - 21) that :

$$\alpha = \sqrt{\frac{2}{\ell(1 - m^2)}} \left(\frac{1 - m}{1 + m} \right)^{m/2} \sqrt{s} + 0(s) \tag{25}$$

therefore, to obtain the expansion of $k_1(s)$, $k_2(s)$ in powers of s, it is necessary to obtain that of $\varphi'(e^{i\beta})$ in powers of α. This is achieved by differentiating equ. (23) with respect to z and expanding it up to the first order in α, using the following changes of variable and function (fig. 4) :

$$z = e^{i\alpha\zeta} \quad ; \quad \varphi'(z) = \ell\, e^{-i\alpha\zeta} [\, U(\zeta) + \alpha\, V(\zeta) + 0(\alpha^2)\,] \tag{26}$$

One obtains thus :

$$U(\zeta) + \alpha\, V(\zeta) = \Gamma + \frac{\overline{\Gamma}'}{2} + \frac{1 - e^{2i\pi m}}{4i\pi} \int_{-1}^{+1} \frac{(\lambda^2 - 1)\, \overline{U(\lambda)}\, d\lambda}{(\lambda - m)(\lambda - \zeta)^2}$$

$$+ \alpha \left[- \frac{i\, \overline{\Gamma}'}{2} (\zeta + m) + \frac{1 - e^{2i\pi m}}{4i\pi} \int_{-1}^{+1} \frac{(\lambda^2 - 1)\, \overline{V(\lambda)}\, d\lambda}{(\lambda - m)(\lambda - \zeta)^2} \right] + 0(\alpha^2) \tag{27}$$

where the interval of integration goes round the point m through the half-plane $Im(\lambda) > 0$.

Identification of the terms of order $\alpha^0 = 1$ in equ. (27) yields an integral equation for U :

$$U(\zeta) = \Gamma + \frac{\overline{\Gamma}'}{2} + \frac{1 - e^{2i\pi m}}{4i\pi} \int_{C^+} \frac{(\lambda^2 - 1)\, \overline{U}(\lambda)\, d\lambda}{(\lambda - m)\, (\lambda - \zeta)^2} \tag{28}$$

where $\overline{U(\lambda)}$ has been replaced by $\overline{U}(\lambda) = \overline{U(\overline{\lambda})}$, and then the integration interval [-1, +1] by the semi-circle C^+ (fig. 4). This equation is identical to that obtained by Wu [1] by a different approach. Once the function U is known, $k_1^* - ik_2^*$ is given by the equation :

$$k_1^* - i\, k_2^* = 2\sqrt{\pi \ell}\; e^{-i\pi m} \left(\frac{1 - m}{1 + m} \right)^{m/2} U(m) \tag{29}$$

which results from the expansion of equ. (22) to the zeroth order in α. References [1] and [7] present numerical calculations of the "components" $F_{ij}(m)$ of the F function based on equs. (28) and (29).

Identification of terms of order α in equ. (27) yields similarly an integral equation for V :

$$V(\zeta) = - \frac{i\, \overline{\Gamma}'}{2} (\zeta + m) + \frac{1 - e^{2i\pi m}}{4\, i\pi} \int_{C^+} \frac{(\lambda^2 - 1)\, \overline{V}(\lambda)\, d\lambda}{(\lambda - m)(\lambda - \zeta)^2} \tag{30}$$

To the best knowledge of the authors, this equation is new. It is of the type $V = V_0 + A\, V$, where the integral operator A is the same as in the equation for U (28). This remarkable property allows for the resolution of equ. (30) by the same method as for equ. (28) [7] : $V(\zeta)$ is given for every $\zeta \in \mathbb{C} - C^+$, by the series

$$V(\zeta) = V_0(\zeta) + A\, V_0(\zeta) + A^2\, V_0(\zeta) + \dots \tag{31}$$

the convergence of which is guaranteed by the fact that the operator A is contracting on the space of functions V analytic on $\mathbb{C} - C^+$ and such that $\lim_{\zeta \to \pm 1} (\zeta^2 - 1)\, V(\zeta) = 0$, with norm defined by

$\| V \| = \sup_{\zeta \in C^-} | (\zeta^2 - 1)\, V(\zeta) |$ (C^- = semi-circle $|\zeta| = 1$, $Im(\zeta) \leqslant 0$).

Once V is known by equ. (31), $k_1^{(1/2)} - i\, k_2^{(1/2)}$ is given by the equation :

$$k_1^{(1/2)} - i\, k_2^{(1/2)} = 2 \sqrt{\frac{2\pi}{1-m^2}} \left(\frac{1-m}{1+m}\right)^m e^{-i\pi m}\, V(m) \qquad (32)$$

which results from equ. (14), the expansion of equ. (22) to the first order in α, and equ. (25).

Since V_0 is proportional to $\bar{\Gamma}' = 1/2\,[\,(N_2 - N_1)\cos 2\gamma + i\,(N_2 - N_1)\sin 2\gamma\,]$, equ. (32) can be put for obvious linearity reasons under the form :

$$\begin{bmatrix} k_1^{(1/2)} \\ k_2^{(1/2)} \end{bmatrix} = \begin{bmatrix} G_1(m) & \tilde{G}_1(m) \\ G_2(m) & \tilde{G}_2(m) \end{bmatrix} \begin{bmatrix} (N_1 - N_2)\cos 2\gamma \\ (N_1 - N_2)\sin 2\gamma \end{bmatrix} \qquad (33)$$

Since $(N_1 - N_2)\cos 2\gamma$ is equal to the non-singular stress T at the tip of the initial crack, the functions G_1 and G_2 are precisely the components of the function G defined by equ. (13). Equ. (13) implies also that the functions $\tilde{G}_1$ and $\tilde{G}_2$ in equ. (33) are zero, though this does not appear clearly in equs. (30-32).

Equations (31) and (32) allow for the numerical computation of functions G_1, G_2, $\tilde{G}_1$, $\tilde{G}_2$ by discretization of the arc C^-, following the same method as in [7]. Functions $\tilde{G}_1$ and $\tilde{G}_2$ are found to be very small (less than the computational error), as predicted by equ. (13). The following table shows the results for the functions G_1 and G_2 :

Angle (°)	0	10	20	30	40	50	60
G_1	0	0.048	0.187	0.402	0.668	0.958	1.238
G_2	0	-0.2734	-0.517	- 0.704	- 0.815	- 0.839	- 0.776

Angle (°)	70	80	90	100	110	120
G_1	1.482	1.662	1.763	1.77	1.70	1.54
G_2	-0.635	-0.434	- 0.196	0.05	0.28	0.48

Angle (°)	130	140	150	160	170	180
G_1	1.31	1.04	0.74	0.45	0.22	$\to 0$
G_2	0.61	0.67	0.66	0.56	0.39	$\to 0$

(The values of G_1 and G_2 for negative angles are given by the following parity relations : $G_1(-m) = G_1(m)$, $G_2(-m) = -\,G_2(m)$).

The zero of the function G_2 is found to be at $m_0 = 97.9°$, which is in excellent agreement with the value given by Bilby and Cardew [10] (98°). Other authors [12] have provided an approximate calculation of functions G_1 and G_2, but the exact calculation shows this approximation to be rather inaccurate.

3.2 - Lemma : Asymptotic form of $\left[\frac{dk}{ds}(s)\right]_{\text{straight}}$:

Before identifying functions H, K, L, M, we need a lemma giving the asymptotic form of $\left[\frac{dk}{ds}(s)\right]_{\text{straight}}$ for $s \to 0$. This quantity is defined as the derivative of the stress intensity factors with respect to the crack length at the point s if the crack, extended up to s, is further extended in a straight manner (fig. 5).

If the crack is extended in a straight manner in the direction πm from the point $s = 0$, the stress intensity factors along this extension have the asymptotic form (by equs. (14), (13) and (17) with $a^* = 0$ and $C^* = 0$) :

$$\left[k(s)\right]_{\text{straight}} = k^* + T\,G(m)\sqrt{s} + \left[k^{(1)}\right]_{\substack{\text{straight}\\ \pi m}} s + o(s) \qquad (34)$$

Since this expansion contains a term proportional to $\sqrt{s}$,

$$\left[\frac{dk}{ds}(s{=}0)\right]_{\text{straight}} = \lim_{s \to 0} \frac{[k(s)]_{\text{straight}} - k^*}{s}$$

(fig. 5) is infinite ; therefore $\left[\frac{dk}{ds}(s)\right]_{\text{straight}}$ tends towards infinity when s tends towards 0.

To precise the behaviour of $\left[\frac{dk}{ds}(s)\right]_{\text{straight}}$, let us define, for every extension starting from $s = 0$ in the initial direction πm :

$$\bar{k}(s) = k(s) - T\,G(m)\sqrt{s} \qquad (35)$$

On the straight extension in the direction πm, one has by equ.(34) :

$$\left[\bar{k}(s)\right]_{\text{straight}} = k^* + \left[k^{(1)}\right]_{\substack{\text{straight}\\ \pi m}} s + o(s) \qquad (36)$$

$\left[\frac{d\bar{k}}{ds}(s)\right]_{\text{straight}}$ being defined in the same way as $\left[\frac{dk}{ds}(s)\right]_{\text{straight}}$, equ. (36) implies that $\left[\frac{d\bar{k}}{ds}(s{=}0)\right]_{\text{straight}} = \lim\limits_{s \to 0} \frac{[\bar{k}(s)]_{\text{straight}} - k^*}{s}$ is finite and equal to $[k^{(1)}]_{\substack{\text{straight}\\ \pi m}}$. Therefore :

$\left[\frac{d\bar{k}}{ds}(s)\right]_{\text{straight}} = \left[k^{(1)}\right]_{\substack{\text{straight}\\ \pi m}} + o(1)$. By the definition of $\bar{k}(s)$

(equ. (35)), this means that

$$\left.\begin{aligned}\left[\frac{dk}{ds}(s)\right]_{straight} &= \left[\frac{d\bar{k}}{ds}(s)\right]_{straight} + \frac{T\ G(m)}{2\sqrt{s}} \\ &= \frac{T\ G(m)}{2\sqrt{s}} + \left[k^{(1)}\right]_{\substack{straight \\ \pi m}} + o(1)\end{aligned}\right\} \quad (37)$$

3.3 - Identification of the functions H, K, L, M :

These functions describe the effect of the curvature of the crack extension on the expansion of the stress intensity factors (see equs. (13) and (17)). To obtain them, we will model the curved crack extension as a series of n straight segments delimited by nodes 0, 1, ..., n, of length s/n, calculate the stress intensity factors at the last node n, and let n tend towards infinity (fig.6). The kink angles α_1, ..., α_{n-1} at the nodes 1, ..., n-1 are adjusted in such a way as to reproduce the shape $v = a^* u^{3/2} + C^*/2\ u^2 + o(u^2)$ of the crack extension (fig. 6) :

$$\left.\begin{aligned}\alpha_p &= \frac{dv}{du}\ (\text{node } p) - \frac{dv}{du}\ (\text{node } p-1) = \\ &\frac{3}{2}\, a^* \sqrt{\frac{p\ s}{n}} - \frac{3}{2}\, a^* \sqrt{\frac{(p-1)\ s}{n}} + C^* \frac{s}{n} + o(s)\end{aligned}\right\} \quad (38)$$

The stress intensity factors at the node p + 1 (before the kink) are deduced from those at the node p by equ. (14), which reads with obvious notations

$$k_{p+1} = k_p^* + k_p^{(1/2)} \sqrt{\frac{s}{n}} + k_p^{(1)}\ \frac{s}{n} \quad (39)$$

(higher order terms are disregarded because their contribution in $k(s) = k_n$ is $0(s^{3/2})$).

k_p^* is given by equ. (7) : $k_p^* = F\ (\alpha_p/\pi,\ k_p)$. Since $\alpha_p = 0(\sqrt{s})$ and the expression of k_p^* is needed only up to the term proportional to s, it is sufficient to use, in this equation, the expansion of F to the 2nd order in m given e.g. in [12] or below (§ 4.2). One gets thus using equ. (38) :

$$\left.\begin{aligned}k_{p,1}^* = k_{p,1} - \frac{9}{4}\, a^* \left(\sqrt{\frac{p}{n}} - \sqrt{\frac{p-1}{n}}\right) k_{p,2} \sqrt{s} - \frac{3}{2}\frac{a^*}{n}\, k_{p,2}\ s\ - \\ \frac{27}{32}\, a^{*2} \left(\sqrt{\frac{p}{n}} - \sqrt{\frac{p-1}{n}}\right)^2 k_{p,1}\ s\end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned}k_{p,2}^* = k_{p,2} + \frac{3}{4}\, a^* \left(\sqrt{\frac{p}{n}} - \sqrt{\frac{p-1}{n}}\right) k_{p,1} \sqrt{s} + \frac{a^*}{2n}\, k_{p,1}\ s\ - \\ \left(\frac{27}{32} + \frac{9}{\pi^2}\right) a^{*2} \left(\sqrt{\frac{p}{n}} - \sqrt{\frac{p-1}{n}}\right)^2 k_{p,2}\ s\end{aligned}\right\} \quad (41)$$

$k_p^{(1/2)} \sqrt{\frac{s}{n}}$ is given by equ. (13) with $a^* = 0$:

$$k_p^{(1/2)} \sqrt{\frac{s}{n}} = T_p \; G\left(\frac{\alpha_p}{\pi}\right) \sqrt{\frac{s}{n}} ,$$

where T_p is the non-singular stress at the node p (before the kink). $G_1(0)$ and $G_2(0)$ being 0 (see above) and α_p being given by equ.(38), this can be written as $\frac{3}{2} \frac{a^*}{\pi} (\sqrt{p} - \sqrt{p-1})\; T_p \; G'(0) \; \frac{s}{n}$, disregarding terms of order $o(s^{3/2})$; thus $k_p^{(1/2)} \sqrt{\frac{s}{n}}$ is in fact of order $o(s)$. T_p can be replaced by the non-singular stress T^* just after the kink (the error introduced being $o(s)$), which can be shown (by techniques analogous to those of section 2) to be equal to $T \tilde{F}(m)$ where $\tilde{F}$ is some universal function. We get thus :

$$k_p^{(1/2)} \sqrt{\frac{s}{n}} = \frac{3}{2} \frac{a^*}{\pi n} (\sqrt{p} - \sqrt{p-1}) \; T \; \tilde{F}(m) \; G'(0) \; s \qquad (42)$$

$k_p^{(1)} \frac{s}{n}$ is given by equ. (17) with $a^* = 0$ and $C^* = 0$:

$$k_p^{(1)} \frac{s}{n} = \left[k_p^{(1)} \right]_{\substack{\text{straight} \\ \alpha_p}} \frac{s}{n} .$$

In the quantity $\left[k_p^{(1)} \right]_{\substack{\text{straight} \\ \alpha_p}}$, α_p can be replaced by 0 and the previous succession of segments by the real, regular extension because the errors introduced tend towards 0 when n tends towards infinity ; this means replacing

$$\left[k_p^{(1)} \right]_{\substack{\text{straight} \\ \alpha_p}} \quad \text{by} \quad \left[\frac{dk}{ds} \text{ (node p)} \right]_{\text{straight}} .$$

Using the lemma (equ. (37)), we get thus, up to order $O(s)$:

$$k_p^{(1)} \frac{s}{n} = \frac{T \; G(m)}{2\sqrt{p\,n}} \sqrt{s} + \frac{1}{n} \left[k^{(1)} \right]_{\substack{\text{straight} \\ \pi m}} s \qquad (43)$$

(which shows that $k_p^{(1)} \frac{s}{n}$ is in fact of order $\sqrt{s}$).

Using equs. (39 - 43), $k_{p+1,1}$ and $k_{p+1,2}$ are expressed as linear combinations of $k_{p,1}$, $k_{p,2}$, T, $[k_1^{(1)}]_{\substack{\text{straight} \\ \pi m}}$ and $[k_2^{(1)}]_{\substack{\text{straight} \\ \pi m}}$ which can be put under the matrix form :

$$[X_{p+1}] = \left(1 + [A_p] \sqrt{s} + [B_p] \; s\right) [X_p] \qquad (44)$$

where $[X_p]$ is the 5-dimensional column vector of components $k_{p,1}$,

$k_{p,2}$, T, $\left[k_1^{(1)}\right]_{\substack{\text{straight}\\ \pi m}}$, $\left[k_2^{(1)}\right]_{\substack{\text{straight}\\ \pi m}}$ and where $[A_p]$ and $[B_p]$ are 5 x 5 matrices with components given by :

$$\left.\begin{aligned}
&A_p^{12} = -\frac{9}{4}\,a^*\left(\sqrt{\frac{p}{n}}-\sqrt{\frac{p-1}{n}}\right) \quad ; \quad A_p^{13} = \frac{G_1(m)}{2\sqrt{p\,n}} \quad ;\\
&A_p^{21} = \frac{3}{4}\,a^*\left(\sqrt{\frac{p}{n}}-\sqrt{\frac{p-1}{n}}\right) \quad ; \quad A_p^{23} = \frac{G_2(m)}{2\sqrt{p\,n}} \quad ;\\
&B_p^{11} = -\frac{27}{32}\,a^{*2}\left(\sqrt{\frac{p}{n}}-\sqrt{\frac{p-1}{n}}\right)^2 \quad ; \quad B_p^{12} = -\frac{3}{2}\frac{a^*}{n} \quad ;\\
&B_p^{13} = \frac{3}{2}\frac{a^*}{\pi n}\,(\sqrt{p}-\sqrt{p-1})\;\tilde{F}(m)\;G_1'(0) \quad ; \quad B_p^{14} = \frac{1}{n} \quad ; \quad B_p^{21} = \frac{a^*}{2n} \quad ;\\
&B_p^{22} = -\left(\frac{27}{32}+\frac{9}{\pi^2}\right)a^{*2}\left(\sqrt{\frac{p}{n}}-\sqrt{\frac{p-1}{n}}\right)^2 \quad ;\\
&B_p^{23} = \frac{3}{2}\frac{a^*}{\pi n}\,(\sqrt{p}-\sqrt{p-1})\;\tilde{F}(m)\;G_2'(0) \quad ; \quad B_p^{25} = \frac{1}{n} \quad ;\\
&\text{other } A_p^{ij}\text{'s and } B_p^{ij}\text{'s} = 0\;.
\end{aligned}\right\} \quad (45)$$

Iteration of equ. (44) yields :

$$\begin{aligned}
[X_n] &= \left(1+[A_{n-1}]\sqrt{s}+[B_{n-1}]\,s\right)\;\ldots\;\left(1+[A_1]\sqrt{s}+[B_1]\,s\right)[X_1]\\
&= \left\{1+\sum_{p=1}^{n-1}[A_k]\sqrt{s}\;+\right. \qquad\qquad (46)\\
&\qquad \left.\left(\sum_{p=1}^{n-1}[B_p]+\sum_{p=2}^{n-1}\sum_{q=1}^{p-1}[A_p]\;[A_q]\right)s\right\}[X_1]
\end{aligned}$$

up to order $o(s)$. In addition the stress intensity factors at the node 1 in $[X_1]$ can be replaced by k_1^* and k_2^*, since the error introduced tends towards 0 when n tends towards infinity. Equ. (46) reduces the problem to calculating the limit of certain sums and products of matrices for $n \to \infty$. This is a purely technical matter ; the final result is (skipping mathematical details) :

$$\left.\begin{aligned} k_1(s) = k_1^* + \left\{ T\, G_1(m) - \frac{9}{4}\, a^*\, k_2^* \right\} \sqrt{s} \,+ \\ \left\{ \left[k_1^{(1)} \right]_{\substack{\text{straight} \\ \pi m}} - \frac{9}{8}\, a^*\, T\, G_2(m) - \frac{27}{32}\, a^{*2}\, k_1^* - \frac{3}{2}\, C^*\, k_2^* \right\} s + o(s) \end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned} k_2(s) = k_2^* + \left\{ T\, G_2(m) + \frac{3}{4}\, a^*\, k_1^* \right\} \sqrt{s} \,+ \\ \left\{ \left[k_2^{(1)} \right]_{\substack{\text{straight} \\ \pi m}} + \frac{3}{8}\, a^*\, T\, G_1(m) - \frac{27}{32}\, a^{*2}\, k_2^* + \frac{C^*}{2}\, k_1^* \right\} s + o(s) \end{aligned}\right\} \quad (48)$$

These equations can also be derived by a completely different method based on the requirement of their "self-consistency" (this means that the expansions (47 - 48) around the point 0 must also be valid around points belonging to the regular part of the crack). This derivation cannot be given here for reasons of space.

In the particular case of a straight initial crack and nearly straight propagation, Sumi et al. [4] have given expressions of $k_1(s)$, $k_2(s)$ accurate to the first order in m, a*, C* which are compatible with the (fully general) eqns. (47-48), except for the terms proportional to a* T ; the present results are thought to be more reliable since they can be derived by two independent methods.

4 - PROPAGATION CRITERIA

4.1 - General considerations :

A criterion giving the direction of crack propagation at a generic point 0 will be said to satisfy property (P) if (as suggested by experience) pure mode 1 before the eventual kink is equivalent to no kinking :

$$k_2 = 0 \iff m = 0 \qquad (49).$$

We will show that any self-consistent criterion satisfying (P) must predict, in the general case where $K_2 \neq 0$, a branching angle πm identical to that deduced from the PLS$[k_2^*(k_1, k_2, m) = 0]$. Indeed, as was noted by Cotterell & Rice [3], in its regular part (after the initial kink), a crack propagating according to a criterion verifying (P) is in pure mode 1 : $\forall s > 0, \pi m(s) = 0 \Rightarrow k_2(s) = 0$. Taking the limit $s \to 0$, we get $k_2^* = 0$. Thus application of the criterion after the kink leads to a kink angle identical with that predicted by the PLS. But the kink angle can also be deduced from direct application of the criterion at the kink ; self-consistency demands therefore that these two values of m be identical.

This establishes immediately the inconsistency of some criteria [5,6] which satisfy (P) but differ from the PLS.

We will now consider the case of the more fundamental Griffith criterion (propagation along the direction πm maximizing $G = \frac{1-\nu^2}{E}(k_1^{*2} + k_2^{*2})$).

4.2 - Comparison of the Griffith criterion and the PLS :

The Griffith criterion satisfies (P) (this results from the detailed study of the function $G = G(k_1, k_2, m)$) ; therefore the problem is to determine whether it coïncides with the PLS or not.

The components of the F function being defined as above by $k_i^* = F_{ij}(m)\, k_j$, the kink angle πm predicted by the PLS verifies

$$k_2^* = F_{21}(m)\, k_1 + F_{22}(m)\, k_2 = 0 \qquad (50)$$

that predicted by the Griffith criterion verifies

$$k_1^* \frac{\partial k_1^*}{\partial m} + k_2^* \frac{\partial k_2^*}{\partial m} = 0 \qquad (51)$$

Coïncidence of the two criteria requires that (50) and (51) be verified simultaneously ; then

$$\frac{\partial k_1^*}{\partial m} = F'_{11}(m)\, k_1 + F'_{12}(m)\, k_2 = 0 \qquad (52)$$

is also verified. Thus coïncidence of the two criteria requires that the linear forms defined by (50) and (52) be simultaneously zero, i.e. that

$$F'_{11}(m) / F_{21}(m) = F'_{12}(m) / F_{22}(m) \quad \text{for every } m \qquad (53)$$

Numerical calculations of the functions F'_{ij} s (see [1,2,7,10] show that these two ratios are indeed very close together. However they cannot tell us whether equ. (53) is strictly verified or not ; we will solve this problem by calculating analytically the exact expansion of the functions F'_{ij} s in powers of m.

Using the series expression of U analogous to (31) : $U = U_0 + A\, U_0 + A^2\, U_0 + \ldots$ (see [7]) it is easy to show that :

$$k_1^* - i\, k_2^* = \left(\frac{1-m}{1+m}\right)^{m/2} \sum_{n=0}^{+\infty} x_n(m) \qquad (54)$$

where :

$$x_n(m) = \begin{cases} (k_1 - i\,k_2)\left(\dfrac{\sin\pi\, m}{2\pi}\right)^n e^{-i\pi m}\; X_m^{(n)}\,(m) & \text{if } n \text{ is even} \\ -(k_1 + i\,k_2)\left(\dfrac{\sin\pi\, m}{2\pi}\right)^n X_m^{(n)}\,(m) & \text{if } n \text{ is odd} \end{cases} \quad (55)$$

the $X_m^{(n)}$'s being functions defined on $\mathbb{C} - C^+$ by

$$\left.\begin{aligned} &X_m^{(o)}\,(\zeta) = 1 \quad ; \\ &X_m^{(n+1)}(\zeta) = \int_{C^+} \frac{(\lambda^2 - 1)\,\overline{X}_m^{(n)}\,(\lambda)\,d\lambda}{(\lambda - m)(\lambda - \zeta)^2} \end{aligned}\right\} \quad (56).$$

It can then be shown inductively that the functions $X_m^{(n)}$'s have the following explicit expression :

$$\left.\begin{aligned} &X_m^{(n)}(\zeta) = \sum_{p=o}^{2n}\sum_{q=o}^{n} \frac{p\,!\,a_{pq}^{(n)}(m)}{(\zeta - m)^p}\, P_q^{(n)}\left(-\frac{1}{2\,i\,\pi}\log\frac{\zeta - 1}{\zeta + 1}\right) \quad \text{if } \zeta \neq m \\ &X_m^{(n)}(m) = \sum_{p=o}^{2n}\sum_{q=o}^{n} a_{pq}^{(n)}(m)\,\frac{d^p}{dm^p}\left[P_q^{(n)}\left(-\frac{1}{2\,i\,\pi}\log\frac{m - 1}{m + 1}\right)\right] \end{aligned}\right\} \quad (57),$$

where the logarithm function is defined on $\mathbb{C} - i\,\mathbb{R}^+$ by $\log(\rho\, e^{i\theta}) = \ell n\,\rho + i\,\theta$ with $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$, and where the coefficients $a_{pq}^{(n)}(m)$ and the polynomials $P_q^{(n)}$ are given by the following induction formulae :

$$\left.\begin{aligned} &P_q^{(o)}(X) = X^q \quad ; \\ &P_q^{(n+1)}(X) = (-1)^q \sum_{s=0}^{q} C_q^s\, B_{q-s}\, P_s^{(n)}(X) \end{aligned}\right\} \quad (58)$$

(B_i : i^{th} Bernoulli number [13]) ;

$$a_{oo}^{(o)}(m) = 1 \quad ; \quad \text{other} \quad a_{pq}^{(o)}(m)\text{'s} = 0 \quad (59)\ ;$$

$$\left.\begin{aligned} a_{pq}^{(n+1)}(m) = \frac{2(-1)^{n+1}\,i\,\pi}{q}\Big[&(p - 1)\,\overline{a}_{p,q-1}^{(n)}(m) + \\ &\frac{2m(p-1)}{p}\,\overline{a}_{p-1,q-1}^{(n)}(m) + \frac{m^2 - 1}{p}\,\overline{a}_{p-2,q-1}^{(n)}(m)\Big] + \\ &\frac{2}{p}\,\overline{a}_{p-1,q}^{(n)}(m) \qquad \text{for} \quad p \geqslant 2 \quad \text{and} \quad q \geqslant 1 \end{aligned}\right\} \quad (60)$$

$$a_{1q}^{(n+1)}(m) = 2\ \bar{a}_{oq}^{(n)}(m) \quad \text{for every} \quad q \tag{61}$$

$$a_{oq}^{(n+1)}(m) = \frac{2(-1)^n\ i\ \pi}{q}\ \bar{a}_{o,q-1}^{(n)}(m) \quad \text{for} \quad q \geqslant 1 \tag{62}$$

$$\left.\begin{aligned} a_{po}^{(n+1)}(m) = \frac{2}{p}\ \bar{a}_{p-1,o}^{(n)}(m) + \sum_{r=p}^{2n+2} \sum_{s=o}^{n} \frac{2(-1)^n\, i\pi}{p(s+1)}\ C_{r-2}^{p-2}\, \bar{a}_{r-2,s}^{(n)}(m) \\ \times \frac{d^{r-p}}{dm^{r-p}} \left[(m^2-1)\ P_{s+1}^{(n+1)}\left(-\frac{1}{2\, i\, \pi} \log \frac{m-1}{m+1} \right) \right] \quad \text{for } p \geqslant 2 \end{aligned}\right\} \tag{63}$$

$$a_{oo}^{(n+1)}(m) = 2(-1)^{n+1}\ i\ \pi \sum_{s=o}^{n} \frac{\bar{a}_{os}^{(n)}(m)}{s+1}\ P_{s+1}^{(n+1)}(o) \tag{64}$$

Formulae (54), (55), (57 - 64) allow for the calculation of the expansion of the F_{ij}'s up to an arbitrary order n_0, by applying them with n varying only between 0 and n_0 (since $x_n(m) = O(m^n)$ by equ. (55)) and treating all functions of m as polynomials of suitable order. One obtains thus at the 6th order, after an elementary but very lengthy calculation :

$$F_{11} = 1 - \frac{3\pi^2}{8}\ m^2 + \left(\pi^2 - \frac{5\pi^4}{128}\right) m^4 + \left(\frac{\pi^2}{9} - \frac{11\pi^4}{72} + \frac{119\pi^6}{15360}\right) m^6 + O(m^8)$$

$$F_{12} = -\ \frac{3\pi}{2}\ m + \left(\frac{10\pi}{3} + \frac{\pi^3}{16}\right) m^3 + \left(-\ 2\pi - \frac{133\pi^3}{180} + \frac{59\pi^5}{1280}\right) m^5 + O(m^7)$$

$$F_{21} = \frac{\pi}{2}\ m - \left(\frac{4\pi}{3} + \frac{\pi^3}{48}\right) m^3 + \left(-\ \frac{2\pi}{3} + \frac{13\pi^3}{30} - \frac{59\pi^5}{3840}\right) m^5 + O(m^7)$$

$$F_{22} = 1 - \left(4 + \frac{3\pi^2}{8}\right) m^2 + \left(\frac{8}{3} + \frac{29\pi^2}{18} - \frac{5\pi^4}{128}\right) m^4 + \left(-\ \frac{32}{15} - \frac{4\pi^2}{9} - \frac{1159\pi^4}{7200} + \frac{119\pi^6}{15360}\right) m^6 + O(m^8)$$

which implies that :

$$\frac{F'_{11}}{F_{21}} = -\ \frac{3\pi}{2} + \left(4\pi - \frac{3\pi^3}{8}\right) m^2 + \left(10\pi - \frac{41\pi^3}{30} + \frac{\pi^5}{32}\right) m^4 + O(m^6) \tag{65}$$

$$\frac{F'_{12}}{F_{22}} = -\ \frac{3\pi}{2} + \left(4\pi - \frac{3\pi^3}{8}\right) m^2 + \left(10\pi - \frac{23\pi^3}{18} + \frac{\pi^5}{32}\right) m^4 + O(m^6) \tag{66}$$

Equs. (65- 66) show that equ. (53) is not verified, and thus that the Griffith criterion, though numerically very close to the PLS, does not coïncide with it (1). The reason why the question remained unsolved up to now is that the expansion of the F_{ij}'s was known only up to the 2^{nd} order [12] whereas the present results show that the difference between F'_{11}/F_{21} and F'_{12}/F_{22} appears only when the F_{ij}'s are expanded up to the 6^{th} order.

The general reasoning of § 4.1 shows then that the Griffith criterion is not self-consistent and must therefore be dismissed. Though it is questionable whether this criterion is of a really basic nature, i.e. whether it can be derived rigorously from Griffith's fundamental hypotheses, its rejection generates a serious disturbance in the Griffith theory : indeed, propagation being supposed to occur when G reaches a critical value G_c, it is difficult to admit, from the physical point of view, that propagation occurs in a certain direction where $G = G_c$ whereas G is greater than G_c in a nearby direction.

4.3 - Prediction of crack path :

The reasoning of § 4.1 shows that the PLS is the only possible criterion for purely logical reasons. This criterion implies that k_2 (s) is 0 for every $s > 0$. The successive terms of the expansion (48) of $k_2(s)$ must therefore be equated to 0, which yields values of the geometrical parameters of the crack extension, m, a*, C* :

$$F_{21}(m)\ k_1 + F_{22}(m)\ k_2 = 0 \quad \text{(whence m)} \tag{67}$$

$$a^* = -\frac{4}{3}\ \frac{T\ G_2(m)}{k_1^*} \tag{68}$$

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(1) The possibility of an error in equs. (65 - 66) can be ruled out for the following reasons :
i - Formulae (54),(55), (57 - 64) cannot be erroneous because the development of the F_{ij}'s to the 4^{th} order based on them coïncides with that obtained by another independent method ;
ii - The calculation of the expansion of the F_{ij}'s to the 6^{th} order based on formulae (54),(55),(57 - 64) has been carried out both by hand and on a computer, which eliminates any possibility of an error in the application of these formulae.

$$C^* = -\frac{2}{k_1^*}\left[k_2^{(1)}\right]_{\substack{\text{straight}\\ \pi m}} - \frac{3}{4}\,\frac{a^*\,T\,G_1(m)}{k_1^*} = -\frac{2}{k_1^*}\left[k_2^{(1)}\right]_{\substack{\text{straight}\\ \pi m}} + \frac{T^2\,G_1(m)\,G_2(m)}{k_1^{*2}} \qquad (69)$$

In the case of regular propagation ($m = 0$, $a^* = 0$), the curvature after the point $O(C^*)$ is equal to that before that point (C) and given by :

$$C^* = C = -\frac{2}{k_1}\left[\frac{dk_2}{ds}\right]_{\substack{\text{straight}\\ \pi m = 0}} \qquad (70)$$

where the notation $\left[k_2^{(1)}\right]_{\substack{\text{straight}\\ \pi m = 0}}$ has been replaced by $\left[\frac{dk_2}{ds}\right]_{\substack{\text{straight}\\ \pi m = 0}}$ since the expansion of the stress intensity factors contains no $\sqrt{s}$ term . Equ. (70) may be considered as the general equation of the crack in its regular part. It is recalled that $\left[\frac{dk_2}{ds}\right]_{\substack{\text{straight}\\ \pi m = 0}}$ has no universal expression and must be calculated numerically in each particular case, but (and this is the interest of equ. (70)) depends only on the geometry of the crack before the point O.

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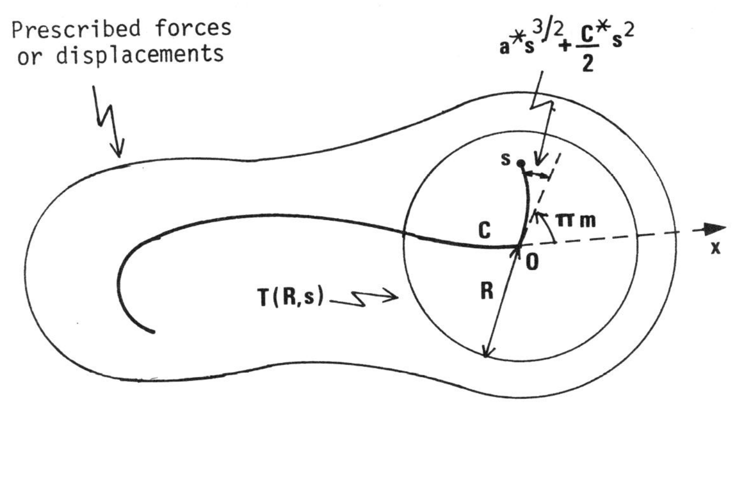

Figure 1 : Curvilinear crack in a two-dimensional body.

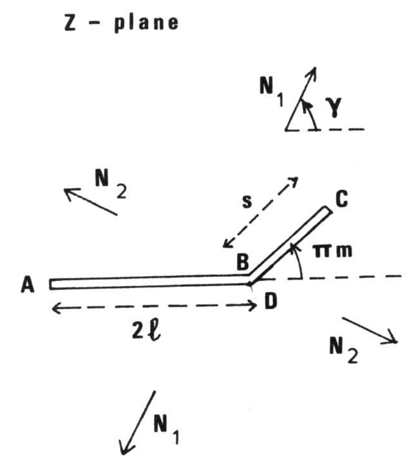

Figure 2 : Straight crack with a straight extension in an infinite body.

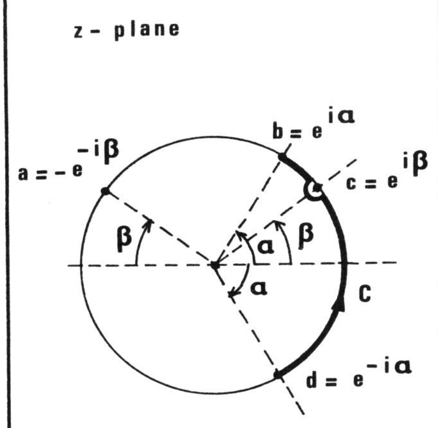

Figure 3 : Geometry of the problem in the z-plane.

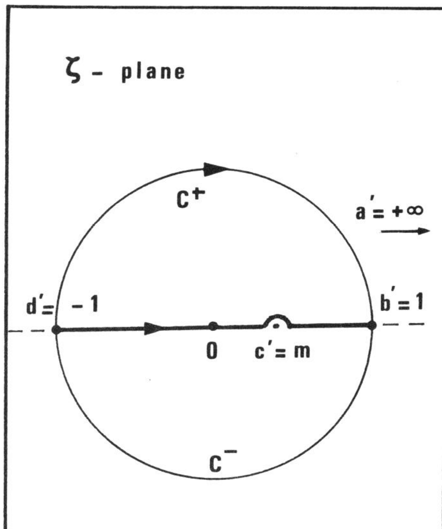

Figure 4 : Geometry of the problem in the ζ-plane.

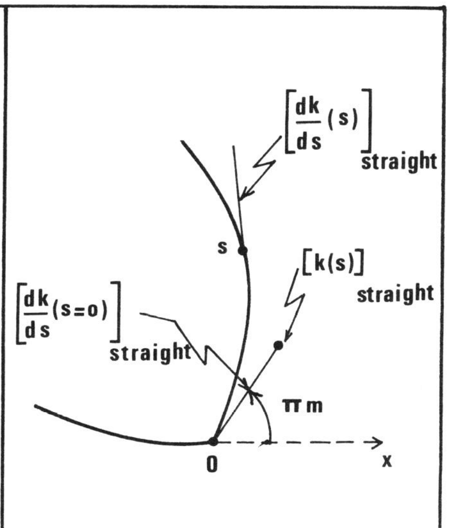

Figure 5 : Definition of some quantities.

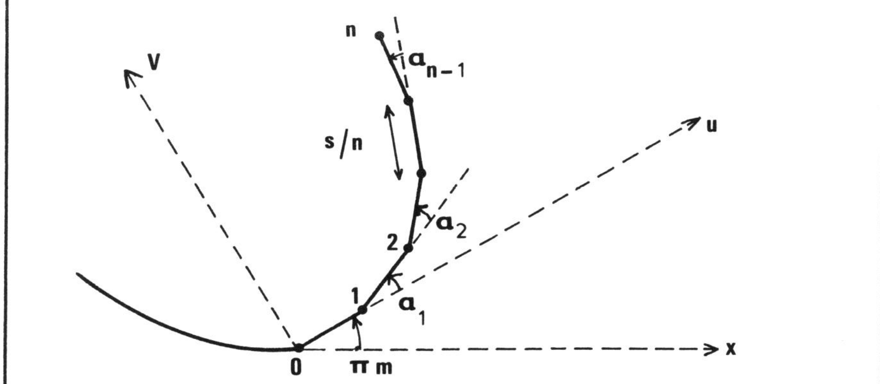

Figure 6 : Schematization of the curved crack extension as a succession of n straight segments.