STABLE GROWTH OF A PENNY-SHAPED CRACK SUBJECTED TO THERMAL LOAD

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ABSTRACT

Equilibrium states which exist between the imposed thermal load and a penny-shaped crack of a given dimension are described quantitatively for the early (stable) phase of fracture growth. Three specific types of thermal loading were considered. The strain field within the non-linear end-zone formed ahead of a quasi-statically growing crack was represented by Wnuk´s model of "final stretch".

Transition to unstable or spontaneous fracture, which follows the quasi-static crack extension, has been predicted for the three configurations considered, i.e., (a) prescribed constant heat flux, (b) given temperature difference across the surface of the crack, and (c)given constant rate of heat extraction due to the effect of a steady-state flow of a cooling liquid. The solution of the latter problem is directly applicable in considerations of stability of a reservoir in a geothermal power generating system.

1. INTRODUCTION

Existence of thermal-stress singularities around the edges of sharp cracks contained in an elastic medium has been demonstrated by the numerous researchers. Olesiak and Sneddon (1960) and then Florence and Goodier (1963) have studied the distributions of thermal stresses arising in the vicinity of a penny-shaped crack which was either disturbing a uniform heat flow (the second paper) or it was subjected to a certain heat flux or a temperature distribution defined across the surface of the penny-shaped crack (the first paper). More recently, the crack opening displacements and the stress intensity factor due to thermal "loadings" of various kinds applied directly to the disc-shaped crack surfaces were investigated by Kassir (1969), Barber (1979), and most recently by Sekine and Mura (1979). Similar investigations were carried out for plane-strain crack disturbing a uniform heat flow by Sekine (1975). In this paper we shall confine attention to the axis mmetrical problem, and in particular, we choose to investigate the loading conditions, i. e., either a prescribed heat flux or a temperature distribution given across the surface of a penny-shaped crack, which eventually lead to an unstable spontaneous enlargement of the crack. If the medium containing the crack were perfectly elastic, one would then deal with the classical Griffith problem involving a transition from a stationary to a catastrophic defect. The novel aspect of our studies, described in the following sections, consists in introduction of non-elastic relaxation zones around the initially sharp edges of the crack, in a fashion analogous to that suggested by Dugdale for ductile metals. Such modification of the crack profile tends to bear out blunting of the crack tip, reduces the infinite stress to a finite value and, most importantly, delays the process of transition into a catastrophic crack propagation. The delay can be explained through a careful study of the preliminary crack extension which is quasi-static and stable up to the point of terminal instability. In imperfectly elastic solids, like ceramics or rocks, this point of transition of fracture mode into spontaneous propagation does not coincide the loading and crack dimension predicted as "critical" from the Griffith criterion of energy balance at fracture. Instead, one arrives at two distinct curves, see Fig. 1; the lower one representing the initiation of the crack growth, occurring at a certain threshold load and initial crack size, while the upper curve represents the locus of terminal instability points at which the propagation of fracture can no longer be controlled. It should be emphasized, though, that at all the intermediate states, which fall between the initiation and the instability loci, crack extension is <u>stable</u> and the load remains in equilibrium with the current crack radius. Load increases monotonically with the crack dimension and a continuous enhancement of load is necessary in order to keep the crack going. One may, there-fore, speak of a crack been "driven" by the external load.

In what follows we shall define the equivalent "load" arising from the thermal conditions imposed on a penny-shaped crack and will discuss the stability of such thermally driven crack. The preliminary crack extension, which is quasi-static, will receive most of the attention. Finally, the instability criterion will be derived. Stable phase of crack growth under mechanical loads has received wide attention. It has been studied in metals by Rice and co-workers (1978-1979), in ceramics by Gerberich (1979), in polymers by McCartney (1979), and in both metals and polymers by this author (1972-1980). In 1972, the concept of "final stretch" was proposed, cf. Wnuk (1072), as a fracture criterion suitable for a description of a quasi-static crack, extension of which in imperfectly elastic materials almost always precedes the onset of unstable failure. This model, which originally was suggested to explain cracking under small scale yielding condition, was eventually extended by Wnuk (1978, 1979), and Smith (1980) to incorporate a situation of large scale yielding (or post-yield rupture). Similar approaches were used for ceramics by Gerberich (1979) and for linear visco-elastic solids by McCartney (1979). Recently, Wnuk and Mura (1980) have applied the final stretch model of quasi-static fracture to a disc-shaped geothermal reservoir contained in a large mass of dry hot rock (Westerly granite).

2. STABLE PHASE OF CRACK GROWTH

The present study expands some of the ideas of the predesessors, the fundemental assumption being that of the existence of a non-linear zone within an elastic medium adjacent to the crack edge, in which due to either microcracking (in rocks and ceramics) or plastic deformation (in metals) a significant stress and displacement redistributions take place. The usual $r^{-1/2}$ singularity in the stresses which open up the crack is removed and replaced by a constant stress, say Y. The strains associated with a quasi-static crack become logarithmically singular at the crack tip and the displacement within the end-zone associated with a slowly moving crack is of the form

$$\left[u_z(r_1,c) \right]_{r_1 \to 0} = u_{tip}(c) - Ar_1 \left[\log (B/r_1) + C\right] + \ldots \quad (2\text{-}1)$$

Here, r_1 denotes the distance measured from the crack tip, c is the crack radius, while A,B, and C are certain functions of the external load, current crack size and the material properties. Two new material properties are suggested by an investigation of the stable phase in crack growth history. These are (a) tearing modulus, say M, and (b) size of the "process zone", say Δ, over which the final act of fracture takes place. Crack is assumed here to move in a sequence of finite "jumps" or "steps" each step being on the order od Δ. Quantity Δ is regarded here to be a microstructural parameter, invariant to the amount of crack growth; it is on the same order of magnitude as the initial COD, at which the blunted crack begins to propagate. For ductile materials Rice (1968) has shown that the region of intensive straining ahead of the crack front is indeed on the same order of magnitude as the initiation COD, say δ_i.

The basic physical assumption underlying the final stretch model of early stages of fracture is that the crack progresses in small but finite steps. The size of each step equals Δ, which is identified with the process zone size, and in case of ceramics or rocks it can be approximated by an average spacing between the microcracks generated ahead of the dominant crack. The requirement that the average strain within the process zone prior to collapse of this zone attains a critical level is equivalent to a condition of a constant crack opening displacement $\hat{\delta}$ (i.e., the "final stretch") generated at a fixed distance Δ from the tip of an advancing crack. Note also that the ratio $\hat{\delta}/\Delta$ is a measure of the crack tip opening angle (CTOA); which has been recognized by many researchers as a quantity invariant to the crack extension. This observation suggests the definition of the tearing modulus M used in the equations given in the sequel, i.e.,

$$M = \left[\pi E/8Y\ (1-\nu^2)\right](\hat{\delta}/\Delta) \tag{2-1a}$$

Both the tearing modulus and the process zone size enter into the non-linear differential equation governing the crack motion, cf. Wnuk (1980)

$$\frac{dR}{dc} = \frac{R}{c} + (\phi')^{-1}\left[M - \phi - \Phi\right]$$

$$R = R\ (c), \phi = \phi\ (R/c), \quad \Phi = \Phi(\Delta, c) \tag{2-2}$$

Symbol R denotes the extent of the end-zone, adjacent to the crack tip within which the stress relaxes to a constant level, Y. The functions* ϕ and Φ can be derived from the expression (2.1) in the following way

$$\phi = \left[\pi E/4\ (1-\nu^2)Y\right]\left[u_{tip}(c)/c\right]$$

$$\Phi = \left[\pi E/4(1-\nu^2)Y\right]\left[\delta u_z(r_1,c)/\delta r_1\right]_{r_1 = \Delta} \tag{2-3}$$

For the contained yielding case the following linear relation between the crack tip opening displacement $u_{tip}(c)$ and the extent of the end-zone ahead of the crack front is valid

$$R(c) = \left|\pi E/4(1-\nu^2)Y\right|\ u_{tip}(c) \tag{2-4}$$

Combining this expression with the definition of the function ϕ, i.e., top equation in (2.3), we notice first that $\phi = R/c$ and then we reduce the differential equation governing slow growth of the penny-shaped crack (2.2) as follows

$$\frac{dR}{dc} = M - \Phi\ (\Delta, c) \tag{2-5}$$

This equation is valid only within the small scale yielding range. Let us now apply the formulae given above to a penny-shaped crack opened up by certain thermal loadings, some of which eventually lead to a catastrophic fracture. In particular, we shall consider three loading conditions

(a) Constant heat flux across the surface of the crack, such that the temperature gradient $d\theta/dz$ is positive, if r, z and ϑ are the cylindical coordinates.

(b) Constant temperature difference between the crack surface and the surrounding medium, say $\theta = -\theta_o$.

(c) Heat flux is a prescribed function of the radial distance r over the crack surface. The distribution considered here is applicable in the studies of stability of a geothermal reservoir subjected to so called "strain controlled" loading system.

All of the conditions described above imply cooling effect present at the crack surface and hence the tendency of the induced thermal contraction of the surrounding material to increase the gap between the opposite surfaces of the initial defect. A situation of this kind prevails for example in a geothermal reservoir in which the hot mass of rock is continually cooled through circulation of water within a disc-shaped vertical crevass. Other applications of practical importance involving use of non-homogeneous reinforced materials in severe thermal environments may be pointed out. Note that for the first two cases considered we deal with a positive K-gradient situation, while the third case will be shown to reduce to a negative K-gradient configuration. As we shall show for cases (a) and (b) the derivative $dK/dc>0$, which implies a tendency to unstable crack extension. The state of spontaneous crack propagation, however, is preceded by the sequence of equilibrium states in which the external load (i.e., temperature difference or heat flux) remains in equlibrium with a crack of a specific size, see Fig. 1. The functional relation between the loading parameter λ and the equilibrium (stable) crack dimension is a priori unknown and it will be subject to determination. For an infinite medium surrounding a penny-shaped crack the loading parameter λ is related to the crack radius c through the following differential equation

$$\frac{d\lambda}{dc} = \frac{M - \Phi - (\lambda^2/2)}{c\lambda} \qquad (2\text{-}6)$$

in which both ϕ and Φ are certain expressions which for the geometrical configuration considered here can be deduced from the equations given by Wnuk (1980). The non-linear differential equation (2.6) defines the

the quasi-statically extending crack, i.e., it defines the curve $\lambda=\lambda(c)$ shown schematically in Fig.1.

3. NEAR-TIP DISPLACEMENT FIELD FOR A QUASI-STATIC CRACK

The curve labeled "2" in Fig. 2, which represents the displacement outside the crack circumference, i.e., for $r > c$, $z = 0$, has been drawn according to the solution given for a quasi-static crack derived from the concept of the final stretch, cf. Wnuk (1980). Restricting the range of application of these solutions to the small scale yielding situation we obtain

$$[u_z(r_1,c)]_{r_1 \to 0+} = (Y/P_{eq})w_o\{R-r_1[\log(\frac{8c}{r_1})-(\frac{c}{2R})]\} + \ldots \quad (3\text{-}1)$$

where the constant w_o is defined by

$$w_o = 4P_{eq}(1 - \nu^2)/\pi E \quad (3\text{-}2)$$

equivalent (thermal) load $\lambda=(P_{eq}/Y)$ which remains in equilibrium with

Equation (3.1) suggests two definitions; that of the crack tip opening

$$u_{tip}(c) = [4Y(1 - \nu^2)/\pi E]\, R(c) \quad (3\text{-}3)$$

and that of the function $\Phi(\Delta,c)$, i.e.,

$$\Phi(\Delta,c) = \frac{1}{2}\log(8c/\Delta) - (c/2R) \quad (3\text{-}4)$$

Now we can specify the right-hand-side of the governing equations (2.5) and (2.6). Thus for an enlarging penny-shaped crack we obtain (i) the apparent fracture toughness

$$\frac{dR}{dc} = M + \frac{c}{2R} - \frac{1}{2}\log(8c/\Delta), \quad R = R(c) \quad (3\text{-}5)$$

and (ii) the equilibrium thermal load

$$\frac{d\lambda}{dc} = \frac{2M - \log(8c/\Delta) + (\lambda^2/2)}{2c\lambda}, \quad \lambda = \lambda(c) \quad (3\text{-}6)$$

We shall now discuss the implications, particularly those concerned with the crack stability, as they follow from the equations (3.5) and (3.6). Before such discussion can be made meaningful we ought to define the equivalent thermal loads for the three cases considered, i.e., for (a) constant heat flux, (b) constant temperature, and (c) a prescribed heat flux across the surface of the crack. In order to be able to do that we have to derive the stress intensity factors for the three types of boundary conditions.

Some of the integral curves obtained numerically from eq. (3.6) are shown in Fig.3. It should be observed that while the curves $\lambda=\lambda(c)$ drawn for the cases a and b indicate a transition from stable to unstable fracture propagation (the transition points are marked by small circles), there is no such feature in the load/crack length curve obtained for the third type of loading considered here. Omitting some algebraic details we shall give the appropriate forms defining the dimensionless thermal loads for the three cases. They are as follows:

(a) For a prescribed heat flux Q_o across the crack surface, the loading parameter, $\lambda = (1/2)\sqrt{\pi c}(K/Y)$ reads

$$\lambda^a = \frac{\pi E \ \alpha c \ Q_o}{8Y(1 - \nu)} \tag{3-7}$$

in where α denotes the coefficient of linear thermal expansion of the solid containing the crack. The terminal instability locus is predicted from the requirement of a vanishing derivative, $d\lambda/dc = 0$. It follows then

$$\frac{c_f}{\Delta} = \frac{1}{8} \exp\left\{ \frac{2 - m_f^4 \ (c_f/\Delta)^4}{m_f^2 \ (c_f/\Delta)^2} + 2M \right\} \tag{3-8}$$

in where symbol m denotes the non-dimensional heat flux, $m = [(\pi E\alpha Q_o \Delta)/8Y(1-\nu)]$, while the index "f" is added to emphasize the occurrence of the unstable fracture at the point (c_f, m_f). The pair (c_f, m_f), i.e., the critical crack radius and the critical load, may be determined from eg. (3.8) in a numerical way.

(b) When the surface of a penny-shaped crack is maintained at a constant temperature lower that the temperature of the suroounding medium, say

$$\theta = - \theta_o, \quad 0 < r < c, \quad z = 0 \tag{3-9}$$

and when the temperature gradient outside the crack is zero across the symmetry plane, i.e.,

$$\frac{\partial \theta}{\partial z} = 0, \quad r > c, \quad z = 0 \tag{3-10}$$

the nondimensional loading parameter which enters the diferential equation (3.6) assumes the form

$$\lambda^b = \frac{E \alpha \theta_o}{6Y (1-\nu)} \tag{3-11}$$

Temperatures at the crack growth initiation and at the transition point to a rapid propagation are predicted as follows

$$(\theta_o)_{ini} = \frac{3\pi(1-\nu)}{E\alpha} \frac{K_{Ic}}{\sqrt{\pi c_o}} \tag{3-12}$$

and

$$\frac{c_f}{\Delta} = \frac{1}{8} \exp \{ \frac{2 - n^4 (\theta_{of})^4}{n^2 (\theta_{of})^2} + 2M \} \tag{3-13}$$

Here the constant n is defined by

$$n = \frac{E \alpha}{6Y(1-\nu)} \tag{3-14}$$

(c) When the surfaces of the crack are subjected to a liquid flowing in and out the crack, and if the process occurs at a constant heat extraction rate, Q_{eff}, the K-factor associated with the stress field generated around the circumference of such a crack may be shown to be inversely proportional to the square root of the crack radius, i.e.

$$K^{(c)} = \frac{E \alpha Q_{eff}}{8(1-\nu) \sqrt{\pi c}\, k_r} \tag{3-15}$$

Here, k_r denotes the thermal conducticity of the rock. The derivative $\partial K/\partial c$ is, therefore, negative and the crack extension process is inherently stable, as it is seen in Fig. 4 . Loading parameter for this case is related to the surrent crack length and the heat extraction rate Q_{eff} (which in turn is a certain function of crack radius!), in the following way:

$$\lambda^c = \frac{E \alpha Q_{eff}}{16Y(1-\nu) c k_r} \tag{3-16}$$

The plot of λ^c vs. the current crack radius, shown in Fig. 4, implies an uninterrupted equilibrium between the thermal load Q_{eff} and the crack size. Since the derivative dQ_{eff}/dc never becomes negative, fracture in this case is always stable.

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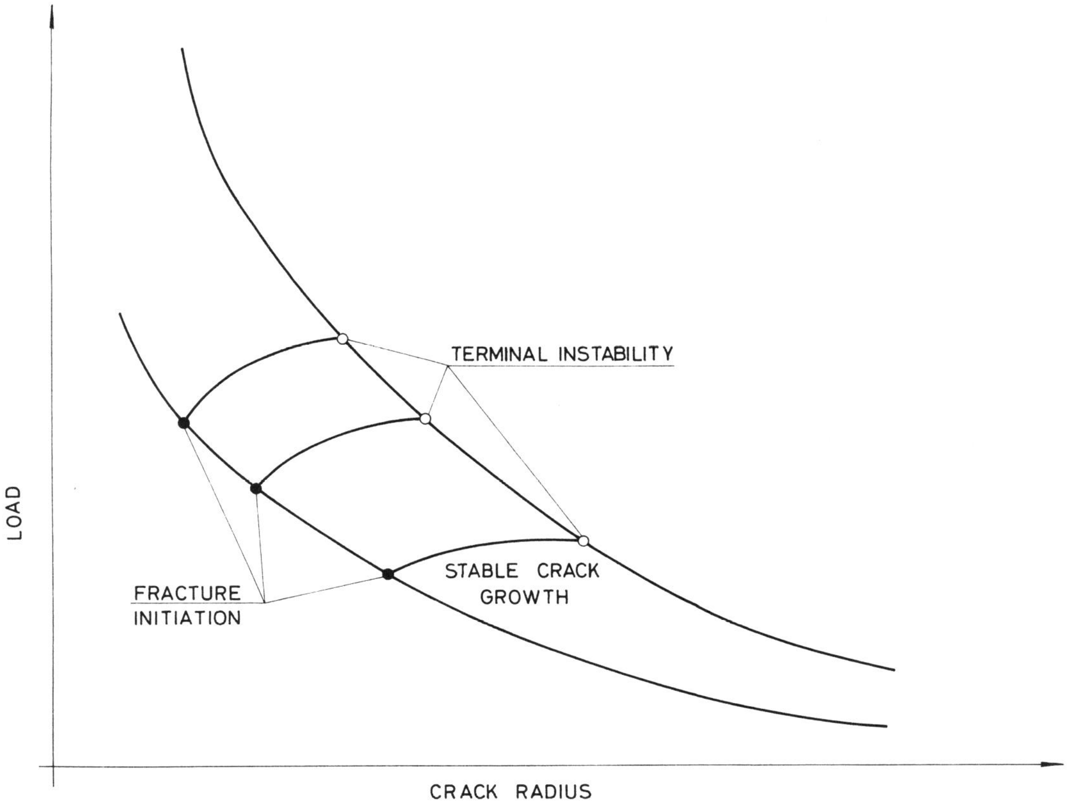
LOAD
TERMINAL INSTABILITY
FRACTURE
INITIATION
STABLE CRACK
GROWTH
CRACK RADIUS

Fig. 1

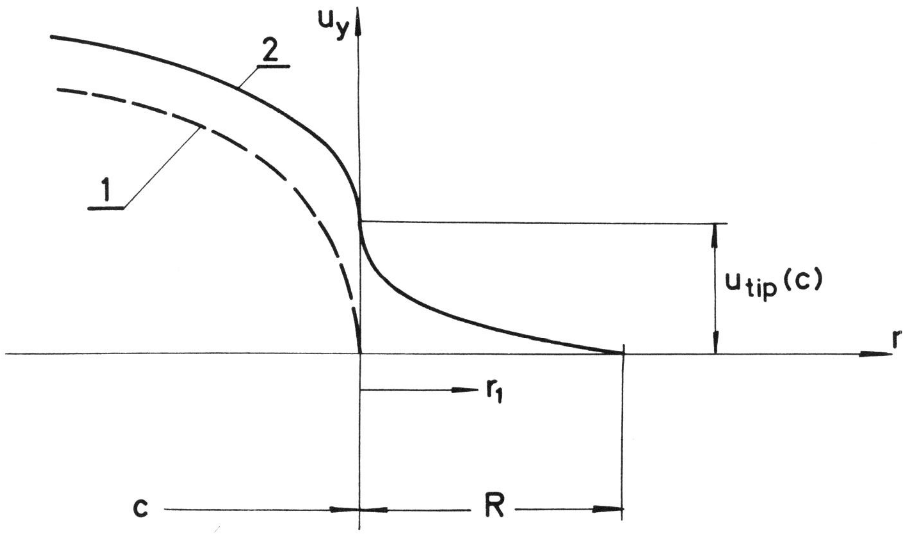
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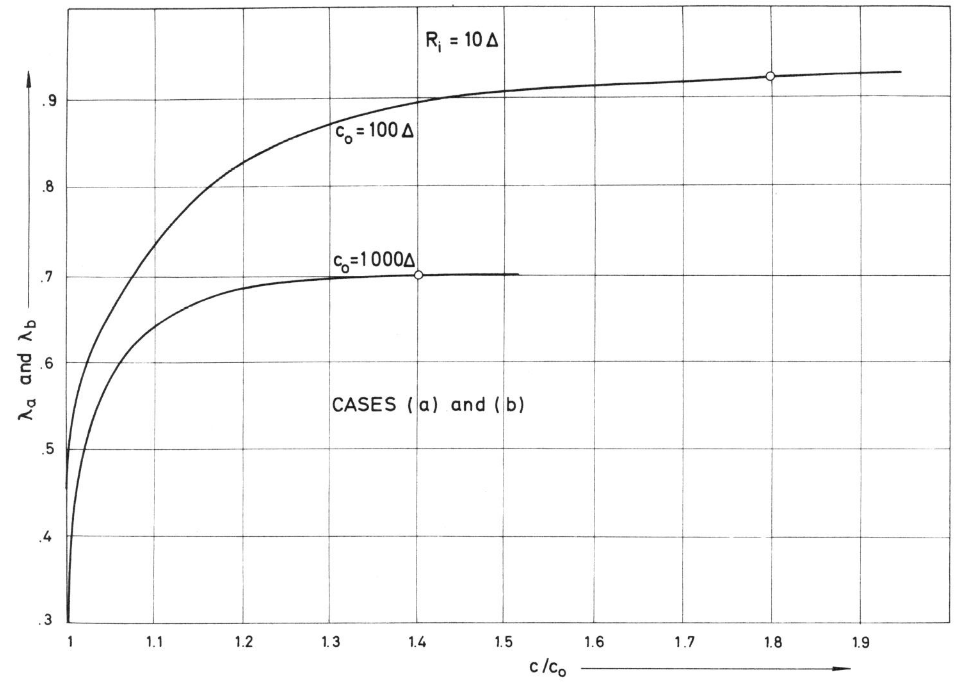
Fig. 2


R_i = 10 Δ
c_o = 100 Δ
c_o=1000Δ
CASES (a) and (b)
λ_a and λ_b
.3
.4
.5
.6
.7
.8
.9
1
1.1
1.2
1.3
1.4
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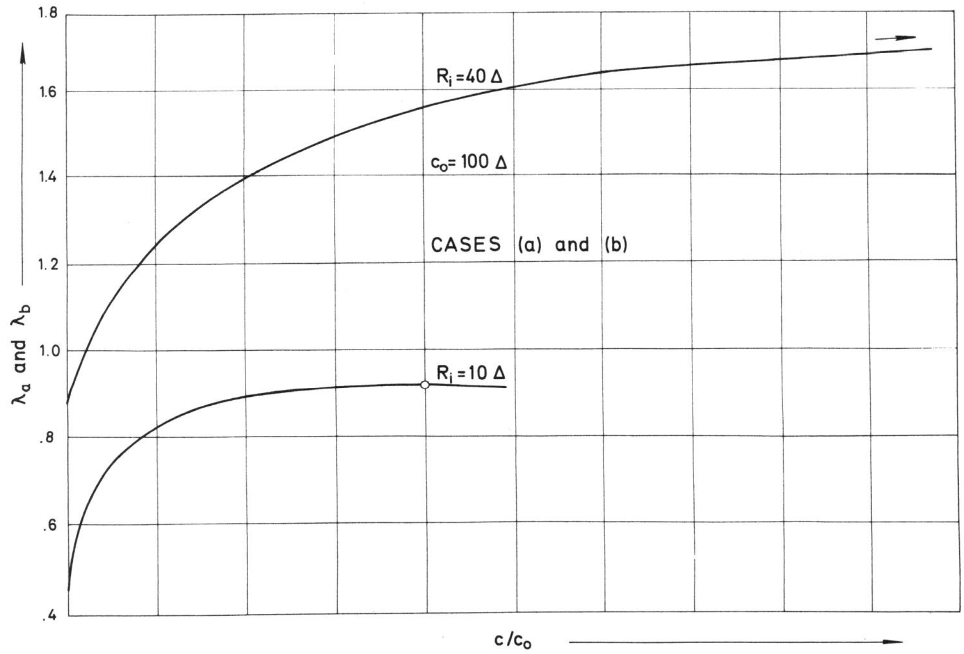
1.7
1.8
1.9
c/c_o

Fig. 3a


R_i =40 Δ
c_o= 100 Δ
CASES (a) and (b)
R_i =10 Δ
λ_a and λ_b
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.6
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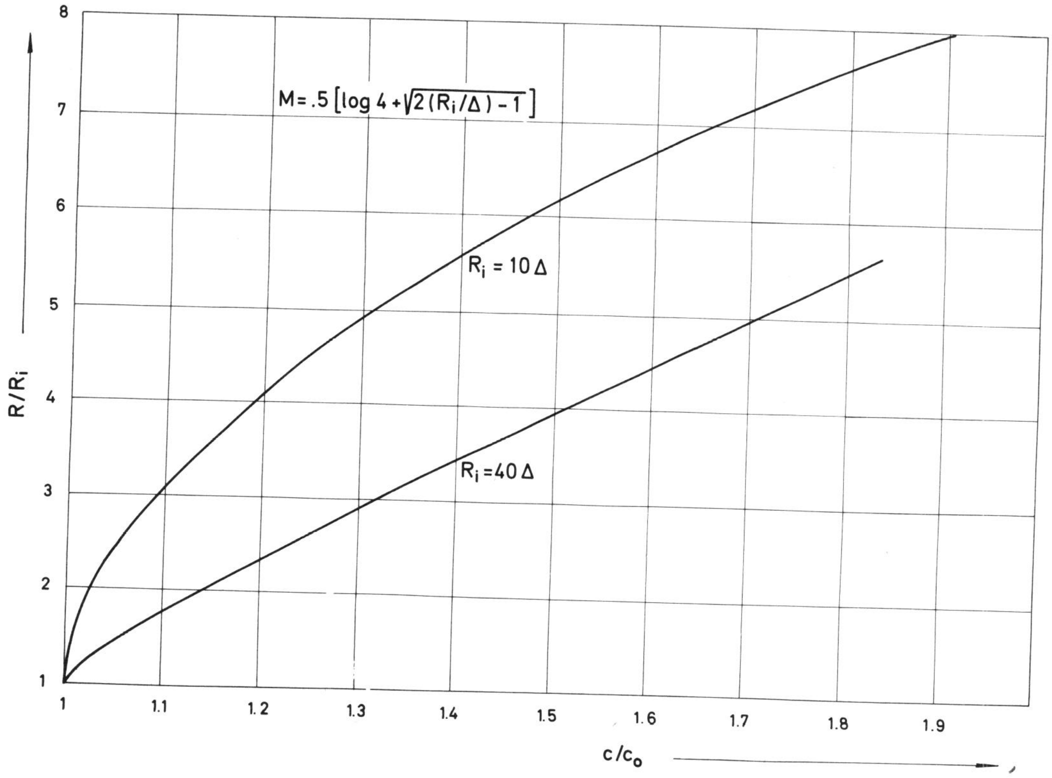
1.0
1.2
1.4
1.6
1.8
c/c_o

Fig. 3b


M=.5 [log 4 + √2(Ri/Δ) - 1]
Ri = 10 Δ
Ri = 40 Δ
R/Ri
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6
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1.5
1.6
1.7
1.8
1.9

Fig. 4