1. SOME PROPERTIES OF THE BOUNDARY LAYER IN ELASTIC BODIES CONTAINING A CRACK

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Introduction

Consider an elastic body with a straight fronted plane through-crack of length 2c, loaded by a uniformly distributed stress σ_∞ perpendicular to the crack plane (F i g. 1). If such a body has an infinitely large thickness, the problem is one of plane strain, and the stress and strain field in the neighbourhood of the crack is described two-dimensionally by small deformation theory of elasticity. According to this theory, at the crack front the stresses and the strains become infinitely large and consequently the solution fails.

If in addition a load-free lateral surface is introduced, the problem becomes three-dimensional. The plane strain solution still applies to the portion of the body remote from the surface, but does not to a layer of material adjoining the lateral surface, which causes a perturbation to the original two-dimensional configuration.

Since this perturbation quickly fades with increasing distance z from the surface, a typical boundary layer problem is raised. Within this boundary layer there is a transition from a plane stress situation at the surface to a plane strain situation in the interior. The lateral contraction near the crack tip and the formation of shear lips, phenomena familiar from testing fracture mechanics specimens, indicate the existence of a boundary layer.

This paper presents some information on the lateral boundary layer, based on the evaluation of a mechanical model describing the z-variation of the out-of-plane stress σ_z.

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Description of the model

In [1] a mechanical model is described, which allows calculation of the z-variation of σ_z, the stress component in the direction of the crack front, for the kind of problem shown in F i g. 1. A detailed explanation can be found there. Here only a brief discussion of the basic idea is given in order to supply the reader with a basis for judging its results, which are subsequently used to define the boundary layer.

The volume of the body near to the crack front, which is affected by the stress concentration, is divided into two cylindrical elements. They are bounded by contours of constant sums of the in-plane stresses σ_r and σ_φ, and their length equals the thickness 2t of the body (F i g. 2).

Average values of $(\sigma_r+\sigma_\varphi)$ over the cross sections of the elements are derived from the near tip solution for the two dimensional stress field. This is possible because the area under the σ_r,σ_φ versus r curve is finite, although the stresses become infinite at r = 0.

If the two elements were free to deform independently, they would suffer different lateral contractions due to the difference in the sums of the averaged in-plane stresses and remain free of transverse stresses. Continuity of the whole body may be preserved, if stresses σ_z and τ_{rz} are introduced. Observing the boundary conditions at the lateral surface as well as at the plane of symmetry, z=t, and proposing a function which describes the retroaction of σ_z on σ_r and σ_φ, average values for σ_z and τ_{rz} may be calculated as functions of z.

The model describes the mechanism of transverse constraint correctly. Since it uses average in-plane stresses as an input, it can only be expected to yield average values of the transverse stresses. This is an inherent limitation which is not uncommon in engineering stress analysis. The usefulness of the results depends entirely on a reasonable choice for the size of the elements. The method described herein was successfully checked against experimental results, which were very difficult to obtain to the required accuracy.

The radius $b(\varphi)$ of the outer element may be chosen as the fading length of the stress concentration and the radius of the inner element as a fixed fraction of it, i.e. as $\mu.b(\varphi)$. Then, by varying μ, information can be gained about the variation of σ_z and τ_{rz} with the distance r from the crack front, since it is possible, for any value of μ, to calculate approximately the distance r at which the average stress over the inner element is equal to the local stress.

Definition of the boundary layer

The model described in the previous paragraph gives an idea of the three-dimensional stress field in deeply notched bodies of finite thickness, if a solution of the corresponding two-dimensional problem is available. In the case that the thickness is large compared to the crack length, the solution assumes a very simple form.

The following formulae give the average stresses σ_z in and τ_{rz} around a small element of radius μb:

$$\sigma_z(z) = \sigma_{z max}\left[1 - (1+\lambda z)e^{-\lambda z}\right] \qquad (1)$$

$$\tau_{rz}(z) = \sqrt{\frac{3\mu}{8(1-\mu)(1+\nu)}}\ \sigma_{z max}\ \lambda z\ e^{-\lambda z} \qquad (2)$$

with

$$\lambda z = \frac{z}{b}\sqrt{\frac{3}{2\mu(1-\mu)(1+\nu)}} \qquad (3)$$

Since the fading length b, which is characteristic of the two-dimensional stress concentration, depends on the geometry of the notch and, in the case of a crack, is directly

proportional to its length, λz is in principle the coordinate z expressed as a fraction of the crack length. In other words, the geometry of the notch (or the crack) enters the model via the spatial extension of the stress concentration.

In formulae (1) and (2) $\sigma_{z max}$ is the maximum possible value of σ_z across an element of fixed size, and this is achieved under fully developed plane strain conditions for the problem considered:

$$\sigma_{z max} = (1-\mu^2)\nu\ \Delta(\sigma_r+\sigma_\varphi)_{max} \qquad (4)$$

In formula (4) $\Delta(\sigma_r+\sigma_\varphi)_{max}$ is the difference in average stress between the two elements of the model, again under a plane strain condition. Poisson's ratio for the material is denoted by ν.

The stresses according to formulae (1) and (2) are shown in F i g. 3 together with a sector of the element on which they act. Since they approach their limiting values asymptotically, the thickness of the boundary layer may be arbitrarily defined by the position, where σ_z attains 95 per cent of its maximum value. In a very thin layer of material adjoining the surface a plane stress situation is closely approximated. The thickness of this sub-layer again may be arbitrarily defined by σ_z reaching 5 per cent of its plane strain value. An experimentally found variation of $\Delta(\sigma_r+\sigma_\varphi)$ within the boundary layer is also included in F i g. 3. It is derived from photoelastic measurements on deeply notched specimens [2] and does therefore not comply with the crack problem treated here.

The thickness of the boundary layer

The thickness of the near plane stress layer ($\sigma_z \leqslant 0{,}05\ \sigma_{z max}$) and the total boundary layer ($\sigma_z \leqslant 0{,}95\ \sigma_{z max}$) in the plane of the crack are plotted versus the element size μ in F i g. 4. On the basis of the assumption that b(0) = c/2, which follows from the two-dimensional solution, the scale on the abscissa may be converted approximately into the relative distance y/c from the crack front. So the increase in thickness of the layers with increasing distance from the crack can be seen.

Some remarks must be made here with respect to the mathematical model: The theory fails near the crack front for the reason mentioned earlier in this paper and at values of $\mu > 0{,}5$ in consequence of assumptions which had to be made for the treatment of the model. Further it should be realized that the stress σ_{zmax} according to formula (4) is not equal to the local plane strain value, which is:

$$\sigma_{z\,plane\ strain} = \nu(\sigma_r+\sigma_\varphi) \qquad (5)$$

Comparing formula (5) to formula (4) shows that the average stress σ_{zmax}, which acts on a finite element, approaches the local plane strain value if μ tends to zero, because then $\Delta(\sigma_r+\sigma_\varphi)_{max}$ approaches $(\sigma_r+\sigma_\varphi)$. This means that the model predicts a true local plane strain situation only in a small region very close to the crack front. With increasing r, σ_{zmax} becomes progressively smaller than $\sigma_{z\,plane\ strain}$.

The thickness of the boundary layer at a particular distance from the crack front in bodies of varying thickness is shown in F i g. 5. The upper diagram shows the thickness d of the total boundary layer as a function of the thickness of the body considered, and as a fraction of the half crack length c. The dimensionless thickness parameters λd and λt are defined by analogy with λz (formula (3)).

It should be noted that with increasing thickness of the body the boundary layer thickness approaches a constant value. The lower diagram shows that portion of the body which is required to complete the transition from a plane stress to a plane strain situation in x,y planes perpendicular to the crack front, again as a function of the body's thickness.

Summary of the conclusions

Bearing in mind the reservations discussed in previous paragraphs, the following conclusions may be drawn:

- The plane stress plane strain transition in a thick body with a through-crack takes place within boundary layers adjoining the load-free lateral surfaces (F i g. 3).
- By the point of inflexion on the σ_z versus z curve a thin near plane stress sublayer is defined (F i g. 3).
- The thicknesses of the boundary layer and the sublayer increase with increasing distance from the crack front (F i g. 4).
- The ratio of the boundary layer thickness at a given distance from the crack front to the crack length decreases with increasing ratio of the body's thickness to the crack length, until it attains a constant minimum value (F i g. 5). That is, according to the model the transition in stress state becomes steeper as the thickness of the body is increased.
- If two bodies have, despite different notch geometries, the same distribution of the in-plane stresses as given by the corresponding two-dimensional solution, and the same thickness, their z-variations of σ_z and τ_{rz} and hence their boundary layer thicknesses are identical.

Final remarks

One of the most important and least understood problems in engineering fracture mechanics is the application of test data from laboratory specimens to full scale structures.

The global load deformation behaviour of a specimen can only be converted into the corresponding behaviour of the structure if both of them have approximately the same dimensions. If this is not the case, a three-dimensional theory is required which accounts for the size effect.

Local fracture events as observed in a test (for instance local initiation of stable crack growth) can be applied more savely to large structures, because the relevant local boundary conditions of the fracture process zone are then more likely to be similar, even if the overall size is different. In order to treat local events, a local failure criterion must be considered which can be written in a general form:

$$\wp(z) > \wp_c(z) \qquad (6)$$

The left side of this criterion represents a mechanical quantity relevant to the expected mode of fracture and involves the solution of the mechanical problem. The right side represents the critical value of this parameter at the occurrence of the fracture event considered. Both sides of equation (6) depend on the local stress state. It has for instance been shown, that the linear elastic stress intensity factor which may be used for $\wp$, varies only weakly across the thickness of a cracked body, because it depends on the in-plane stresses only. See [3] for a comparison of various sources of information. The strong dependence of the fracture behaviour on the specimen thickness cannot be explained on this basis, because the critical value of the stress intensity factor, which appears at the right side of equation (6), is a stronger function of the stress state near to the crack front and this stress state is only completely defined if all three stress components are known and accounted for. A full understanding of fracture, whether local or global, can thus not be achieved without a three-dimensional fracture theory.

List of References

[1] Prantl, G.: A Simple Analytical Model for the Thickness Dependent Transition from Plane Stress to Plane Strain in Bodies with a Crack. Fracture 1977, Vol.3, ICF-4, Waterloo, Canada, pp. 213-221.

[2] Prantl, G.: Die elastischen Querdehnungen und Querspannungen an der Wurzel tiefer Kerben. VDI-Bericht Nr. 297, 1977, pp. 61-66.

[3] Bhandari, S.K., Barrachin, B. and J.L. Picou: On the Three-Dimensional Theories of Cracked Plates. Fracture 1977, Vol. 3, ICF-4, Waterloo, Canada, pp. 361-370.

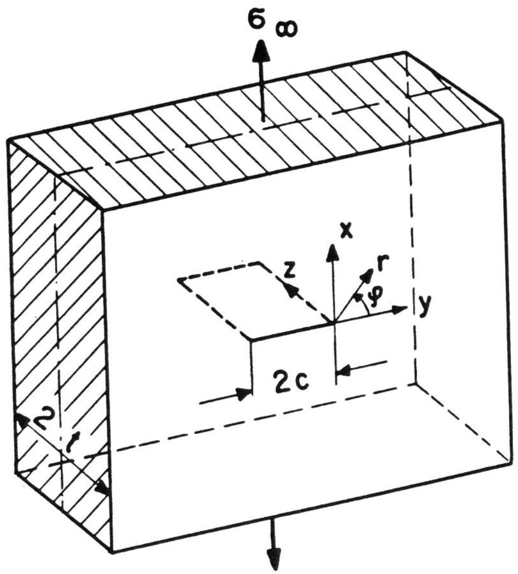


Fig. 1. The Class of Problems Considered and the Coordinates Used

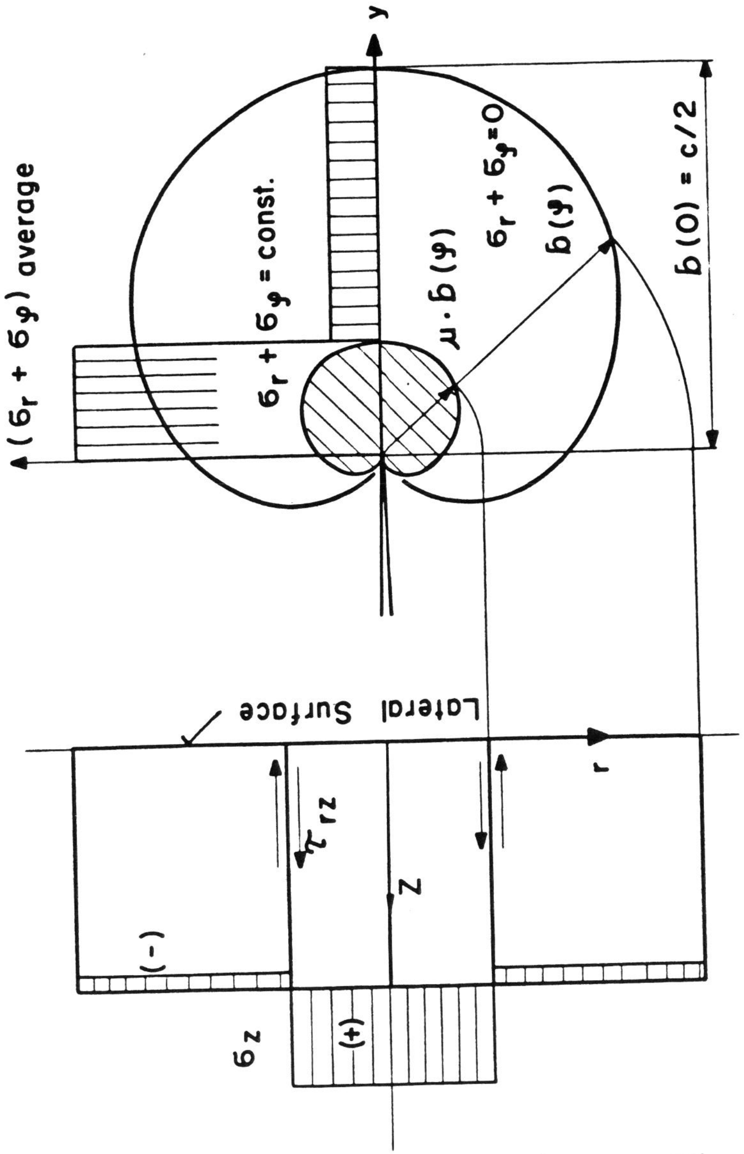
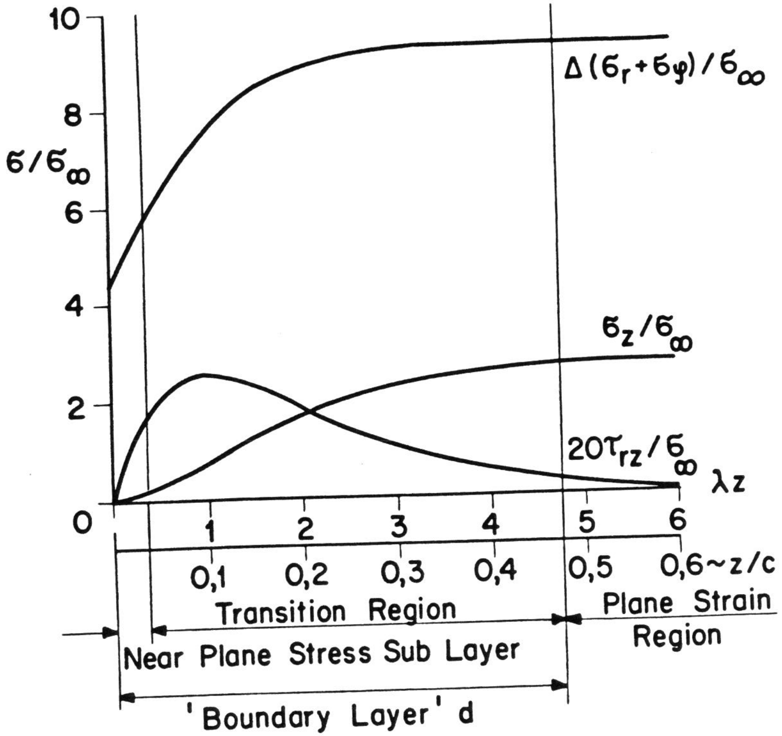
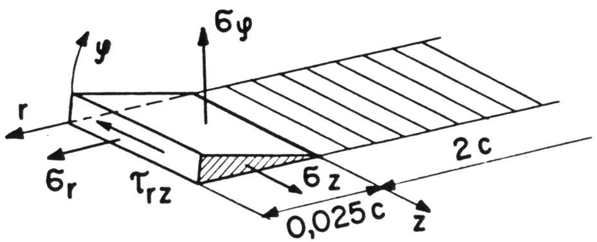


Fig. 2. A Model for the Out-of-Plane Stresses σ_z and τ_{rz}, Based on the Plane Strain Solution for σ_r and σ_φ





$\nu = 0{,}3$, $\mu = 0{,}05$, $t/c = \infty$

Fig. 3. Average Stresses in Front of the Crack in the Boundary Layer of a Thick Body

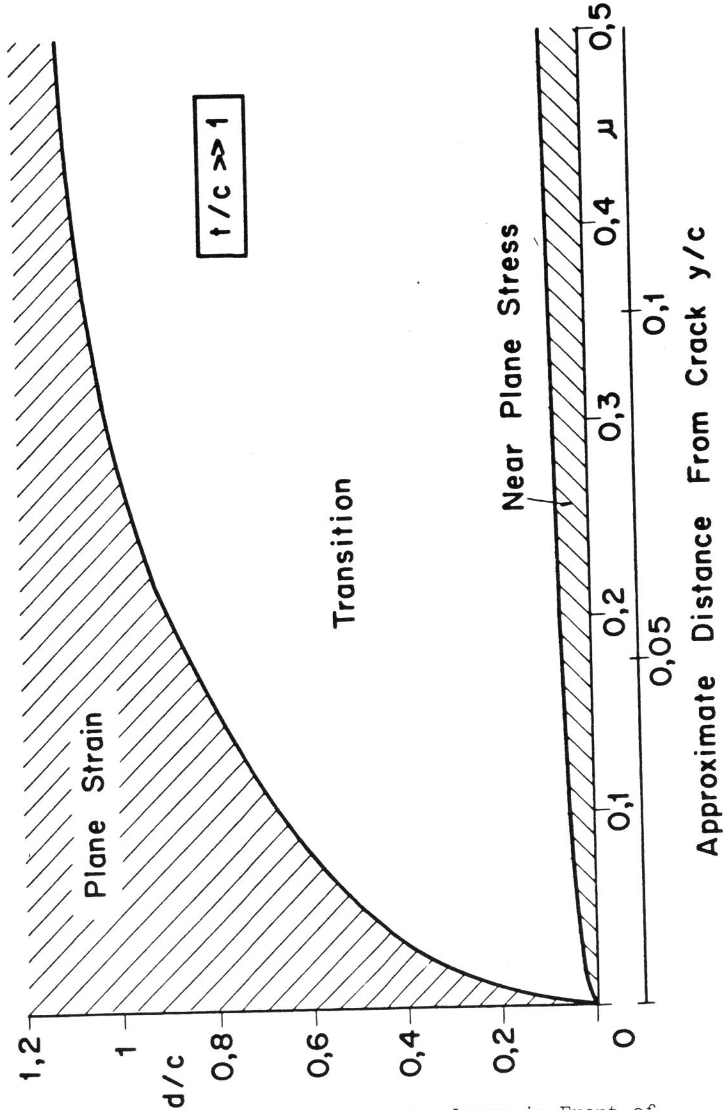


Fig. 4. Thickness of the Boundary Layer in Front of a Crack in a Thick Body

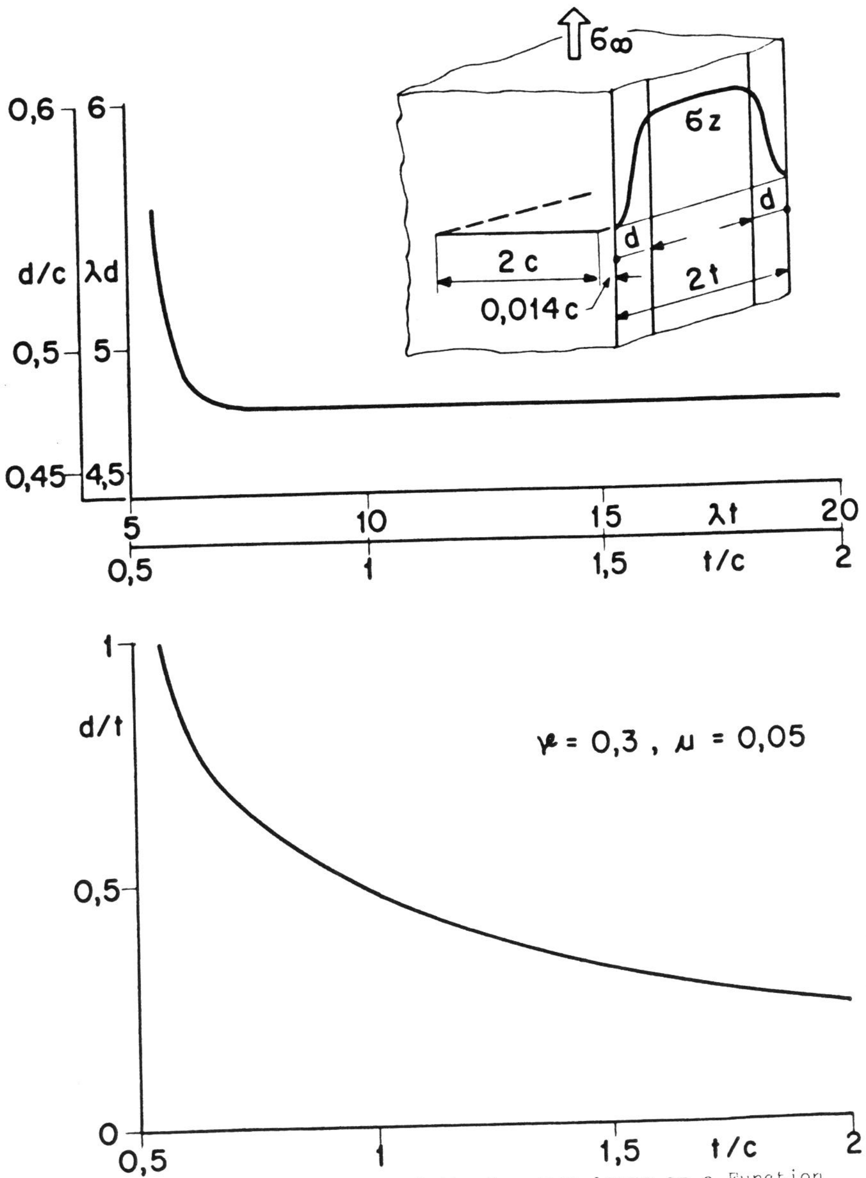


Fig. 5. Thickness of the Boundary Layer as a Function of the Total Thickness