STUDY OF THE DEPENDENCE OF EFFECTIVE COMPLIANCES OF A PLANE WITH AN ARRAY OF ROUND HOLES ON ARRAY PARAMETERS

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Abstract. Regular structure materials are used in different technological processes. Therefore, investigation of the mechanical properties of these materials is of considerable practical interest. These mechanical properties are represented by the relationship between average stresses and effective strains, which can be obtained from the solution of the problem for elastic plane. In this paper, we employ the model of an elastic plane having a biaxial periodic system of round holes to analyze the dependence of the effective elastic parameters on the direction of applied loads and the geometrical characteristics of the system. Parameters anisotropy is demonstrated. The abnormally high values of Poisson's ratio, which are impossible in isotropic media but observed in some anisotropic media, are found.

Introduction

In the current paper the method of multipole expansion is proposed to apply for a problem on an elastic plane containing a bi-axial grid of round holes (see, for example, [1]). Also dependences of effective compliances and Poisson's ratios on the periods parameters (in case of a quadratic grid), and the outer loads' directions are studied, as well as their anisotropy. An existence of the effective Poisson's ratio's values, greater ½, becomes apparent. Such effects are known for some anisotropic materials; see [2-4] for instance. An appearance of longitudinal strains under tangential loads (and vice versa, tangential strains under longitudinal loads) is noted.

Problem statement

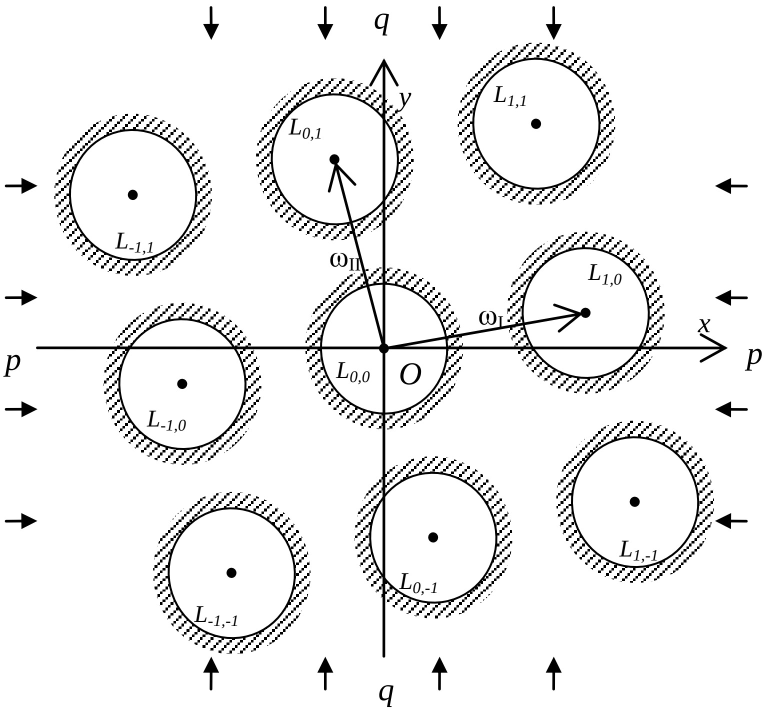

Fig. 1. A bi-axial grid of round holes in an elastic plane

An infinite elastic plane contained the regular system of round holes $L_{l,m}$ is considered (fig. 1). The holes centers form the bi-axial grid with arbitrary periods ω_{I} and ω_{II}. Loads on the holes' contours are zero.
The current study aims to find the effective elastic properties (connections of average stresses with effective strains) and their dependencies on geometric parameters of the grid.

Basic statements
In the work, the method of the multipole expansion is used. The method is explained in [5, 6] in application to the problem of two holes and a regular grid of holes in an elastic plane. It is based on the Kolosov-Muschelishvili potentials [7]. A matter of the method is that the displacement function $g'(t)$ on each of contours is expanded into the power series

$$g'(t) = \sum_{n=-\infty}^{\infty} g_n t^n \tag{1}$$

here t is a local complex coordinate of a point on the contour ($|t| = 1$).
Then the singular boundary integral equation (SBIE) [8, 9] can be transformed into the system of the linear equations of the complex expansion coefficients (their set uniquely defines the strain-stress state of the plane).
By definition, effective compliances of the periodic structure connect the effective strains with the average stresses

$$\tilde{\varepsilon}_{ij} = S_{ijkl}\tilde{\sigma}_{kl} \tag{2}$$

Here one can see that the compliance S_{ijkl} is equal to a strain response $\tilde{\varepsilon}_{ij}$ on a unit effect $\tilde{\sigma}_{kl}$ (the other stresses must be zero). By the way that means that the problem condition must be plain-stress state: $\kappa = (3-\nu)/(1+\nu)$. According to [9], the following expressions are correctly for the free contours problem

$$\tilde{\varepsilon}_{11} + \tilde{\varepsilon}_{22} = \frac{1}{S}\operatorname{Im}\int_L u(t)d\bar{t} + \frac{1-\nu}{E}\left(\tilde{\sigma}_{11} + \tilde{\sigma}_{22}\right) = \frac{1}{S}\operatorname{Im}\int_L u(t)d\bar{t} + 2\frac{1-\nu}{E}\tilde{\sigma} \tag{3}$$

$$\tilde{\varepsilon}_{11} - \tilde{\varepsilon}_{22} + 2i\tilde{\varepsilon}_{12} = \frac{i}{S}\int_L u(t)dt + \frac{1+\nu}{E}\left(\tilde{\sigma}_{11} - \tilde{\sigma}_{22} + 2i\tilde{\sigma}_{12}\right) = \frac{i}{S}\int_L u(t)dt + 2\frac{1+\nu}{E}\left(\tilde{s} + i\tilde{\tau}\right) \tag{4}$$

Here $S = \operatorname{Im}\left(\overline{\omega}_{I}\omega_{II}\right)$ is an area of the basic grid cell.
From

$$\int_L u(t)dt = i\frac{1+\kappa}{2G}\int_L g(t)dt = \frac{4i}{E}\int_L g(t)dt = \frac{4i}{E}\cdot 2\pi i g_{-2} = -\frac{8\pi}{E}g_{-2} \tag{5}$$

$$\int_L u(t)d\bar{t} = \frac{4i}{E}\int_L g(t)d\bar{t} = -\frac{4i}{E}\int_L g(t)\frac{dt}{t^2} = \frac{4i}{E}\cdot 2\pi i g_0 = -\frac{8\pi}{E}g_0 \tag{6}$$

one can get

$$\frac{\tilde{\varepsilon}_{11} + \tilde{\varepsilon}_{22}}{2} = \frac{1-\nu}{E}\tilde{\sigma} - \frac{4\pi}{ES}\operatorname{Im} g_0 \tag{7}$$

$$\frac{\tilde{\varepsilon}_{11}-\tilde{\varepsilon}_{22}}{2}=\frac{1+\nu}{E}\tilde{s}+\frac{4\pi}{ES}\operatorname{Im} g_{-2} \tag{8}$$

$$\tilde{\varepsilon}_{12}=\frac{1+\nu}{E}\tilde{\tau}-\frac{4\pi}{ES}\operatorname{Re} g_{-2} \tag{9}$$

Hence, average strains contain two terms. The first term corresponds uniform strains of the plane without holes under effect of stresses equal to average $\tilde{\sigma}, \tilde{s}, \tilde{\tau}$. The second term is a correction which introduces the holes existence. Note, that the correction is defined by only two terms of the multipole expansion. Moreover, the correction for the average volume strain is defined by the term Im g_0, describing the confining contour deformation; the corrections for the shear components of the average strain are defined by the terms g_{-2}, describing the pure shear strain of the contour.
On the superposition principle, the multipole expansion terms can be presented as a linear combination $\tilde{\sigma}, \tilde{s}, \tilde{\tau}$:

$$g_k = g_k^{\sigma}\tilde{\sigma}+g_k^{s}\tilde{s}+g_k^{\tau}\tilde{\tau} \tag{10}$$

Substituting the combinations Eq. 10 into Eqs. 7-9, the average strains get the following form:

$$\tilde{\varepsilon}_{11}+\tilde{\varepsilon}_{22}=\left(2\frac{1-\nu}{E}-\frac{8\pi}{ES}\operatorname{Im} g_0^{\sigma}\right)\cdot\tilde{\sigma}-\frac{8\pi}{ES}\operatorname{Im} g_0^{s}\cdot\tilde{s}-\frac{8\pi}{ES}\operatorname{Im} g_0^{\tau}\cdot\tilde{\tau} \tag{11}$$

$$\tilde{\varepsilon}_{11}-\tilde{\varepsilon}_{22}=\frac{8\pi}{ES}\operatorname{Im} g_{-2}^{\sigma}\cdot\tilde{\sigma}+\left(2\frac{1+\nu}{E}+\frac{8\pi}{ES}\operatorname{Im} g_{-2}^{s}\right)\cdot\tilde{s}+\frac{8\pi}{ES}\operatorname{Im} g_{-2}^{\tau}\cdot\tilde{\tau} \tag{12}$$

$$\tilde{\varepsilon}_{12}=-\frac{4\pi}{ES}\operatorname{Re} g_{-2}^{\sigma}\cdot\tilde{\sigma}-\frac{4\pi}{ES}\operatorname{Re} g_{-2}^{s}\cdot\tilde{s}+\left(\frac{1+\nu}{E}-\frac{4\pi}{ES}\operatorname{Re} g_{-2}^{\tau}\right)\cdot\tilde{\tau} \tag{13}$$

Here $\tilde{\sigma}=(\tilde{\sigma}_{11}+\tilde{\sigma}_{22})/2$; $\tilde{s}=(\tilde{\sigma}_{11}-\tilde{\sigma}_{22})/2$; $\tilde{\tau}=\tilde{\sigma}_{12}$ are the average stresses components in the grid, $g_n^{\sigma,s,\tau}$ are the expansion coefficients calculating in cases for the corresponding unit average stresses. The effective compliances get the following forms (according to Eq. 2)

$$S_{1122}=\frac{1}{E}\left(-\nu+\frac{2\pi}{S}\left(\operatorname{Im} g_{-2}^{\sigma}-\operatorname{Im} g_0^{\sigma}-\operatorname{Im} g_{-2}^{s}+\operatorname{Im} g_0^{s}\right)\right) \tag{14}$$

$$S_{1112}=S_{1121}=\frac{2\pi}{ES}\left(\operatorname{Im} g_{-2}^{\tau}-\operatorname{Im} g_0^{\tau}\right) \tag{15}$$

$$S_{2211}=\frac{1}{E}\left(-\nu-\frac{2\pi}{S}\left(\operatorname{Im} g_{-2}^{\sigma}+\operatorname{Im} g_0^{\sigma}+\operatorname{Im} g_{-2}^{s}+\operatorname{Im} g_0^{s}\right)\right) \tag{16}$$

$$S_{2222}=\frac{1}{E}\left(1-\frac{2\pi}{S}\left(\operatorname{Im} g_{-2}^{\sigma}-\operatorname{Im} g_{-2}^{s}+\operatorname{Im} g_0^{\sigma}-\operatorname{Im} g_0^{s}\right)\right) \tag{17}$$

$$S_{2212}=S_{2221}=-\frac{2\pi}{ES}\left(\operatorname{Im} g_{-2}^{\tau}+\operatorname{Im} g_0^{\tau}\right) \tag{18}$$

$$S_{1211}=S_{2111}=-\frac{2\pi}{ES}\left(\operatorname{Re} g_{-2}^{\sigma}+\operatorname{Re} g_{-2}^{s}\right) \tag{19}$$

$$S_{1222}=S_{2122}=-\frac{2\pi}{ES}\left(\operatorname{Re} g_{-2}^{\sigma}-\operatorname{Re} g_{-2}^{s}\right) \tag{20}$$

$$S_{1212}=S_{1221}=S_{2112}=S_{2121}=\frac{1}{E}\left(\frac{1+\nu}{2}-\frac{2\pi}{S}\operatorname{Re} g_{-2}^{\tau}\right) \tag{21}$$

Here, E and ν are the elastic parameters of the plane material.
Eqs. 14-21 are used below for the effective compliances calculation.

Effective compliance dependence on the quadratic grid period

In this section a quadratic grid is studied: $\omega_{II} = i\omega_I$.

Values of the effective compliances are calculated for different orientations of the holes grid. In view of the problem symmetry, it is enough to examine the orientation angles from 0° to 45° inclusive. The calculation results are presented in fig. 2 (d is a distance between the neighboring holes: $d = |\omega_I| - 2R$).

In case of $d \geq 10R$, an anisotropy influence on the effective compliance distribution is negligible. Decreasing the distance d, the anisotropy influence begins to behave as a sinusoidal distribution with extrema in the orientation angles 0°/45°, or 22.5° (for the "mixed" compliances like S_{1112}).

If d < 0.5R the effective compliance distribution form changes, a local maximum appears in range ≈ 25°-28° (≈ 17°-20° for S_{1212}) and a local minimum in range ≈ 28°-33° (≈ 12°-17° for S_{1212}). The "mixed" compliances extremum at 22.5° becomes sharper.

The dependences of the effective compliance S_{1111} on the distance d for different grid orientations are presented in fig. 3a. Obviously, the compliance S_{1111} tends to the plane compliance with distance increasing. So, a difference ΔS_{1111} between S_{1111} and the plane compliances is presented in fig. 3b. The log-log plot demonstrates that the dependence $\Delta S_{1111}(d)$ tends to a power-like function $\Delta S_{1111}(d) \approx E^{-1} \cdot 3.22 \cdot d^{-1.67}$. The compliance difference behavior for a diagonal orientation (45°) is the closest form to this function.

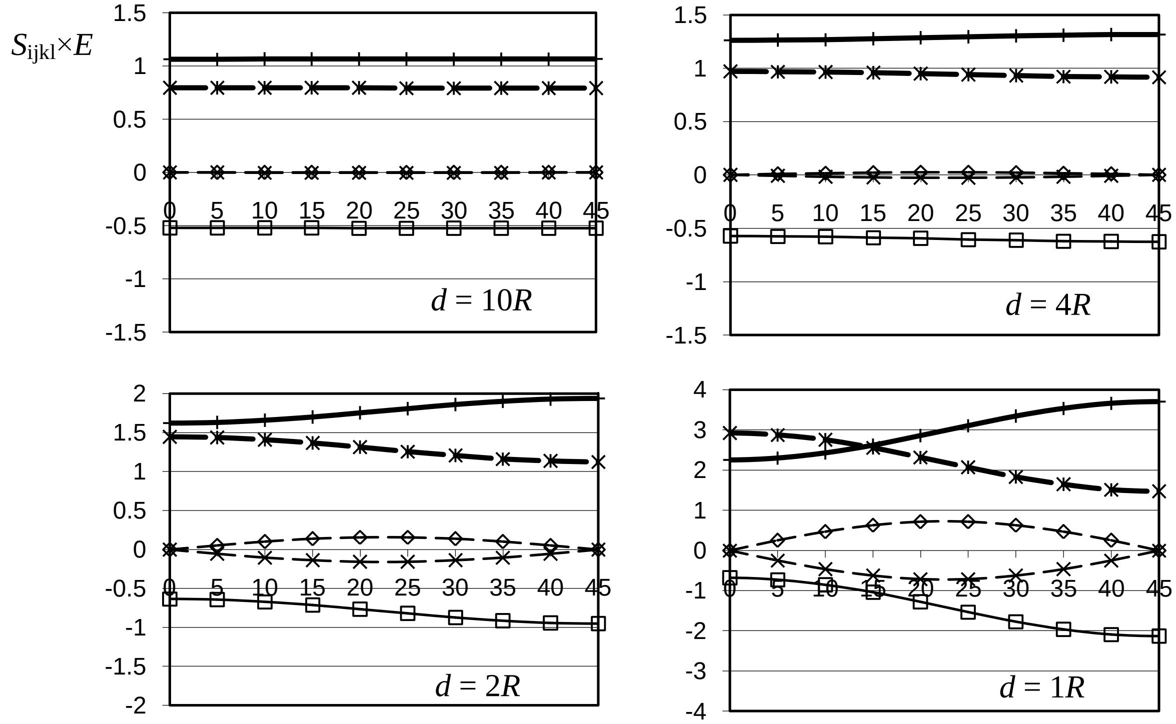

Fig. 2. Effective compliances S_{ijkl} vs. grid orientation angle (°)
for the distances d/R = 10, 4, 2, 1, 0.5, 0.4, 0.3, 0.2, 0.15, 0.1

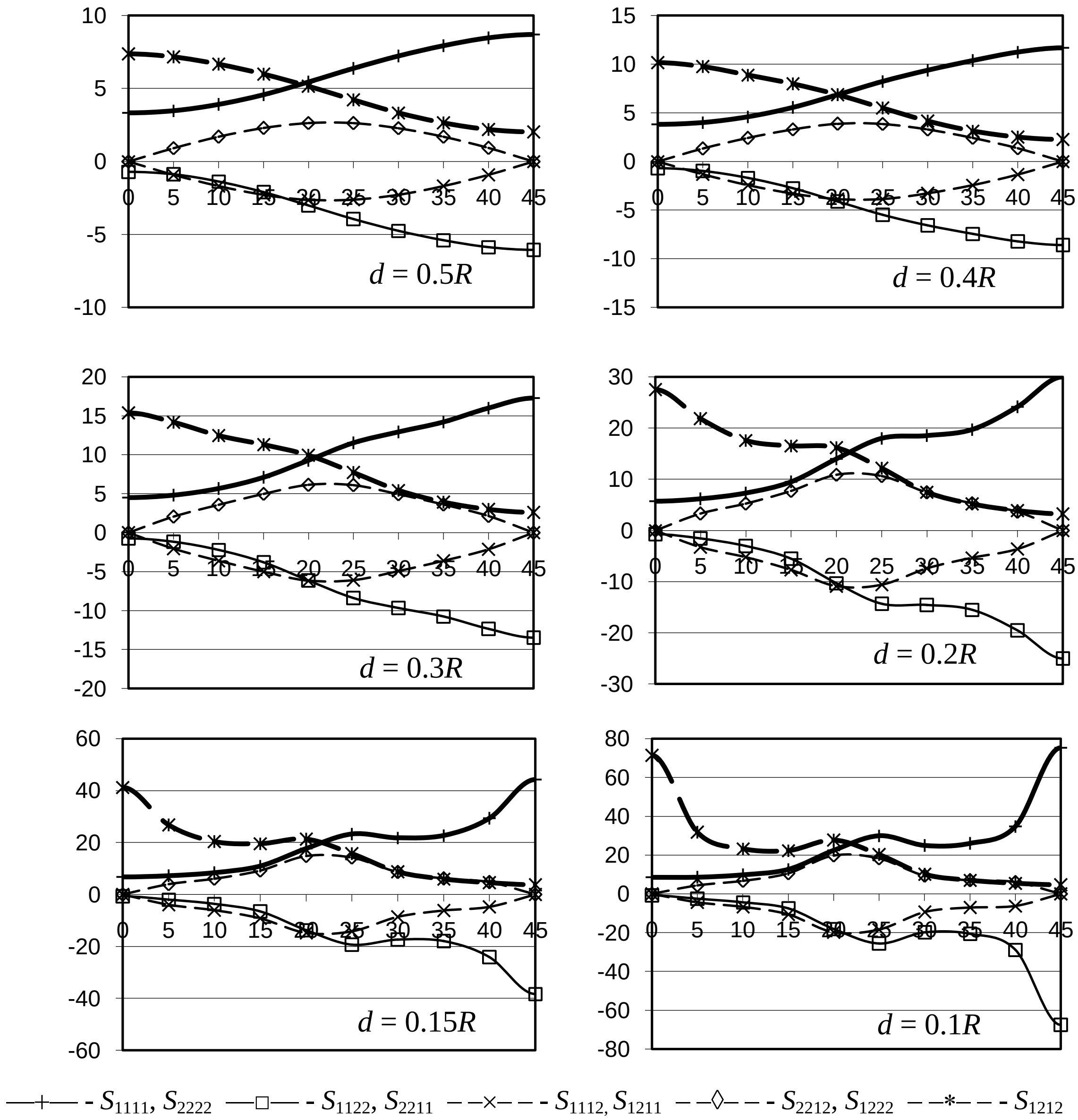

Fig. 2 (continuation). Effective compliances S_{ijkl} vs. grid orientation angle (°) for the distances d/R = 10, 4, 2, 1, 0.5, 0.4, 0.3, 0.2, 0.15, 0.1

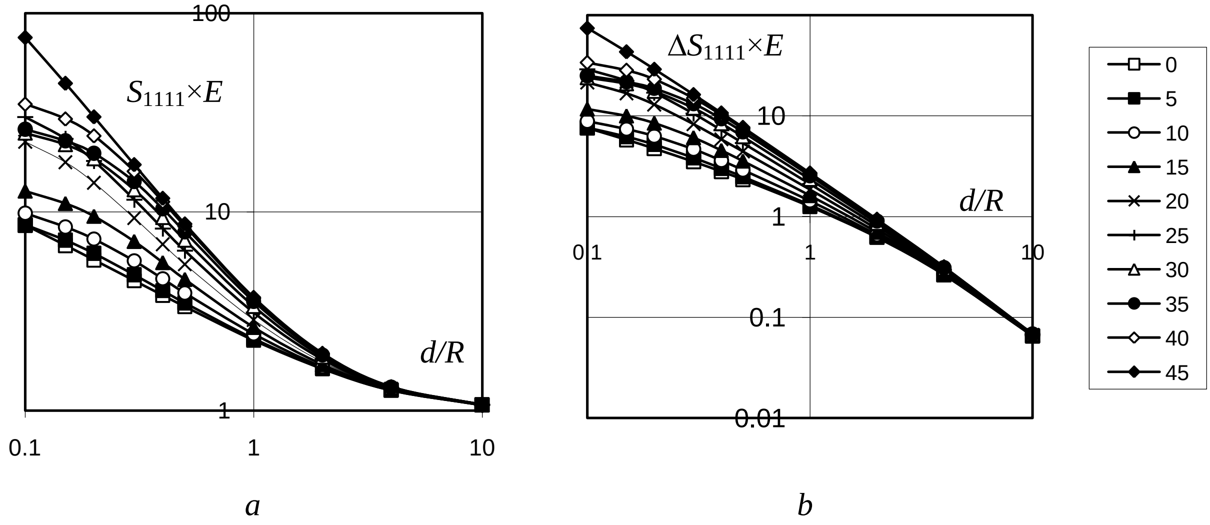

Fig. 3. Dependence of S_{1111} (*a*) and ΔS_{1111} (*b*) on the distance d/R for different grid orientations.

Dependences of an effective Poisson's ratio on the distance *d*

An effective Poisson's ratio $\nu_{eff} = -S_{1122}/S_{1111}$, and its dependence on the plane's Poisson's ratio ν, are studying in this section. Values of ν_{eff} for different ν and different grid geometric parameters are presented in fig. 4.

Note that the effective Poisson's ratio demonstrates a behavior similar S_{1111}. In case of $d > 10R$ the anisotropy influence is insignificant that means ν_{eff} is almost constant and closed to ν. For $0.5R < d < 10R$ the anisotropy manifests itself as sinusoidal dependence on an orientation of the grid; for $d < 0.5R$ the sinusoidal behavior changes, a local maximum appears at orientation values ≈ 24°-26°, and a local minimum near ≈ 33°.

It's worth noting that the influence of the plane's Poisson's ratio ν decreases when the distance *d* increases. Thus the effective Poisson's ratio ν_{eff} becomes substantially depending on a grid orientation.

Plots of dependences of the effective Poisson's ratio ν_{eff} on the distance *d* for ν = 0 and ν = 0.5 are presented in fig. 5. The plots demonstrate that the dependences can have quite complex form. For example, in case of ν = 0.5, at first the effective Poisson's ratio decreases when the distance *d* decreasing. Then, reaching the minimum in range from d/R = 0 to 4, it becomes to grow (excluding the orientation angle 0°). So, if outer loads are known, it is possible to calculate such geometric system parameters as make a transverse strain minimal under the longitudinal stress conditions.

It's important to note that there are wide areas of grid orientation values (on condition $d < 2R$), where the effective Poisson's ratio ν_{eff} exceeds the maximal value 0.5 in isotropic materials. Such a behavior is known for some anisotropic materials. Surveys of materials with anomalous effective Poisson's ratio ($\nu_{eff} > 0.5$ or $\nu_{eff} < 0$), and a theoretic rationale of this effect, can be found in [2-4]. In particular it is demonstrated that a reason is a structural peculiarity of a crystalline lattice of these materials.

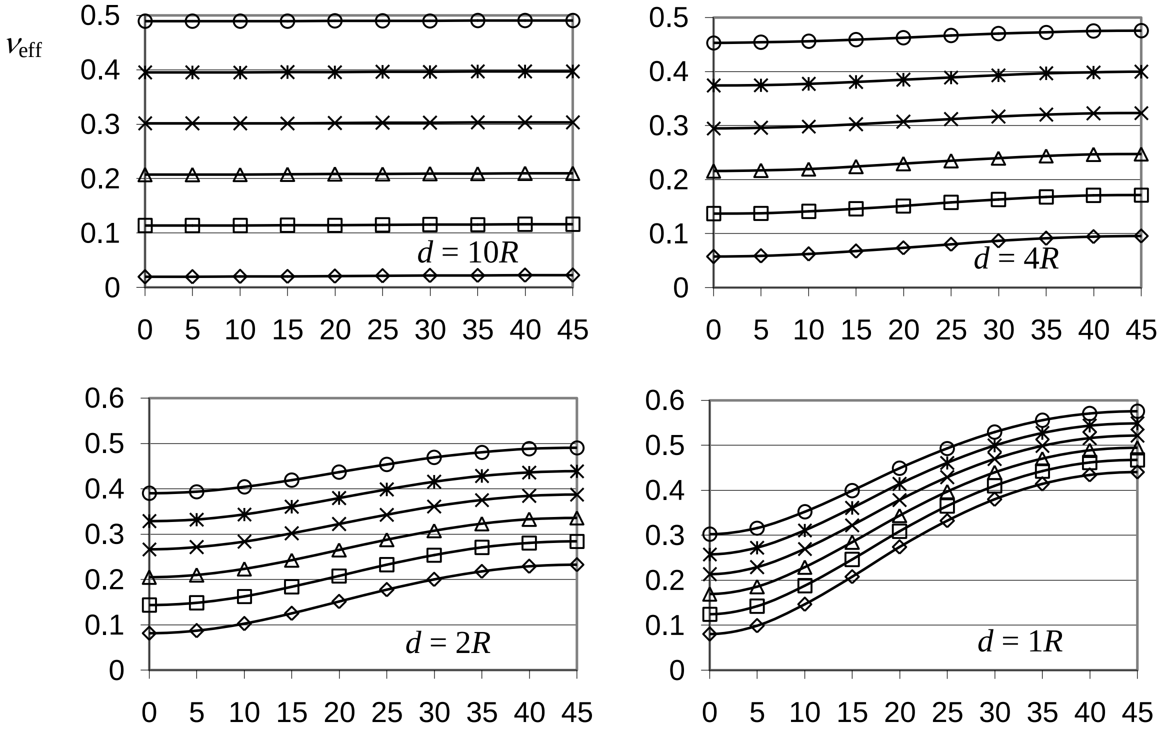

Fig. 4. Effective Poisson's ratio ν_{eff} vs. grid orientation angle (°)
for the distances *d/R* = 10, 4, 2, 1, 0.5, 0.4, 0.3, 0.2, 0.15, 0.1

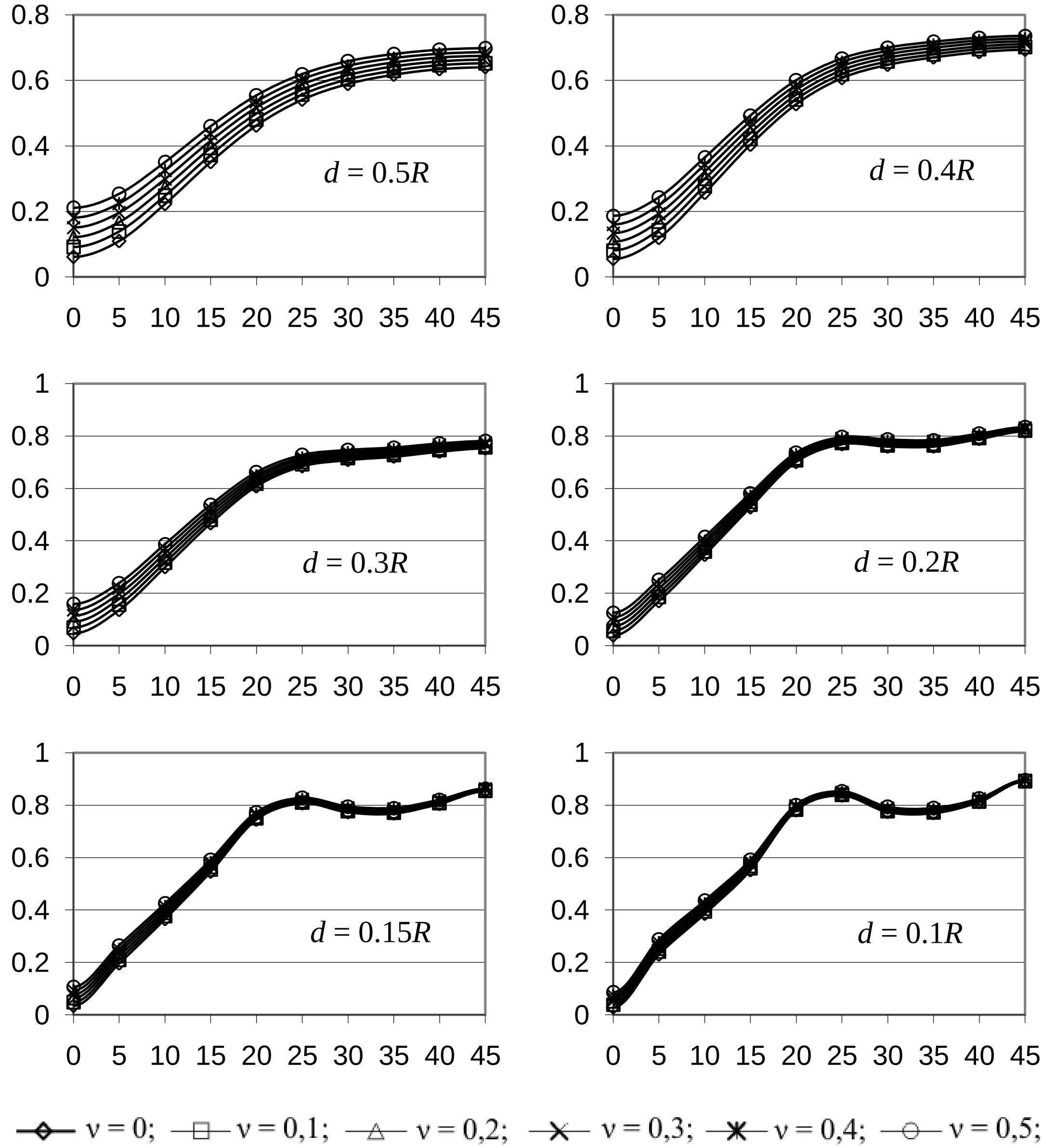

Fig. 4 (continuation). Effective Poisson's ratio ν_{eff} vs. grid orientation angle (°) for the distances *d/R* = 10, 4, 2, 1, 0.5, 0.4, 0.3, 0.2, 0.15, 0.1

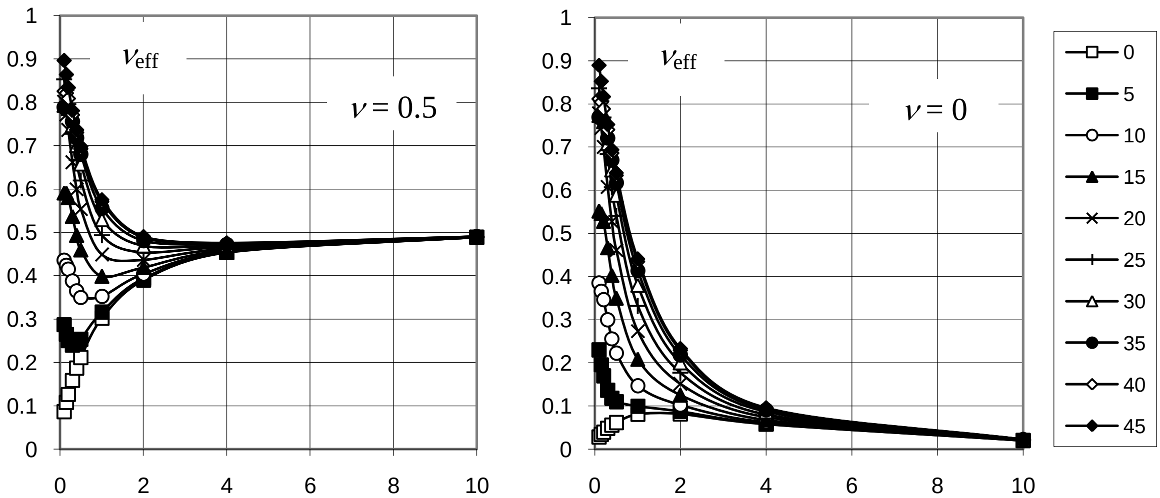

Fig. 5. Dependence of ν_{eff} on the distance *d* for different orientations (ν of the plane is 0.5 and 0)

d/R

Conclusion

A behavior of effective elastic parameters (compliances and Poisson's ratio) of an elastic plane containing a bi-axial system of round holes is studied in the given paper.

Anisotropy of the effective elastic parameters is demonstrated. Three areas of values of a ratio *d/R* are defined, where the anisotropy character differs substantially. In first area ($d/R \geq 10$) the anisotropy is negligible (although the effective elastic parameters can differ from the plane's parameters). In second area ($0.5 \leq d/R \leq 10$) the anisotropy has an expressed sinusoidal kind. In third area ($d/R \leq 0.5$) the sinusoidal kind gets broken; local extrema appear at grid orientations different from longitudinal (0°) and diagonal (45°).

Difference between the effective and plane's longitudinal compliances is studied. The power kind of the dependence of this difference on the ratio *d/R* is shown.

Lowering of the influence of the plane's Poisson's ratio on the effective Poisson's ratio is demonstrated: in case of $d \to 0$, ν_{eff} substantially depends on the grid orientation. It is necessary to note a wide range of grid orientations where $\nu_{\text{eff}} > 0.5$. Such anomalously high values are impossible for isotropic materials but found for some anisotropic materials.

A possible further work development is proposed at two directions:

(1) a study of strength properties (stress concentrations in the material structure);

(2) an extension of the problem into a case of 3D medium [10].

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