Structural-Continual Approach in Fracture Mechanics

Arseny Kashtanov[1,a]

[1] St.-Petersburg State University, 7/9 Universitetskaya emb., St.-Petersburg, 199034, Russia

[a] kashto@mail.ru

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Abstract. The fundamentals of incubation time approach and the sequent structural-continual (kinetic) formulation of incubation time based fracture dynamics are outlined. Exploitation of structural-continuum approach is illustrated on a range of quasi-static and dynamic fracture problems.

Introduction

Attempting to understand the phenomenon of dynamic material strength the scientific community has focused the main efforts on definition of functional strength parameters (dynamic material strength and dynamic fracture toughness) by analogy with quasistatic fracture theory. Nowadays, such approach looks unviable and inconvenient for engineers due to necessity to define material strength functions experimentally for every shape of loading pulse. In 1988 Morozov, Petrov and Utkin have proposed a new criterion of crack initiation under high-rate loading [1] extended to describe the spalling phenomenon in 1990 [2]. It is based on the notion of *incubation time* – the characteristic time of microfracture (relaxation) processes anticipating the macrofracture event. The new incubation time approach become powerful in different problems of fracture dynamics and many peculiarities of dynamic strength phenomenon was soon explained (see, e.g., [3]), including the nature of dynamic branch (corresponded to extremely short loading pulses) of temporal dependence of material strength. The main advantage of *incubation time criterion* of fracture consists in using material constants (depending on no loading rate but on geometrical scale and temperature only) that can be defined from static experiments [1-4]. These material parameters are the static material strength σ_C, the static fracture toughness K_{IC} and the incubation time of fracture τ (defining as time asymptote of dynamic branch of temporal dependence of strength). Later, the incubation time criterion was extended to other dynamic transient processes (listed in Table 1). For these problems the *generalization of incubation time criterion* was proposed [5-8], namely

$$\frac{1}{\tau}\int_{t-\tau}^{t}\frac{1}{d}\int_{x-d}^{x} G_s^{\beta}(x',t')\,dx'dt' \le G_C^{\beta},\quad G_s^{\beta}(t)=\mathrm{sign}\,G(t)\;G^{\beta}(t),\quad d=\frac{2}{\pi}\left(\frac{K_{IC}}{\sigma_C}\right)^2. \qquad (1)$$

Here G is the dynamically changing function characterizing the intensity of external loading (local stress, pressure or stress intensity factor depending on physical process under consideration), G_C represents its critical value under "slow" (quasi-static) loading, is the incubation time (different for distinct physical phenomena) – characteristic time associated with the dynamics of relaxation processes preparing the structural transition in continuum, and d is characteristic size of elementary fracture cell. Parameter β is responsible for loading rate sensitivity and is defined by ductility [8] (in solids) or fluid viscosity [6] (in liquids). Usually it is determined from rather approximation of experimental data on dynamic strength but below we will also provide the method of its determination from quasistatic experiments.

Spalling fracture [5]	$G(t) = \sigma(t)$ – local stress, $G_C = \sigma_C$ – static strength, $\beta = 1$	$\frac{1}{\tau}\int_{t-\tau}^{t} \frac{\sigma(t')}{\sigma_C} dt' \leq 1$
Crack growth initiation [5]	$G(t) = K_I(t)$ – stress intensity factor, $G_C = K_{IC}$ – static fracture toughness, $\beta = 1$	$\frac{1}{\tau}\int_{t-\tau}^{t} \frac{K_I(t')}{K_{IC}} dt' \leq 1$
Dynamic yielding [8]	$G(t) = \sigma(t)$ – local stress, $G_C = \sigma_Y$ – static yield limit, $\beta = 3 \div 40$	$\frac{1}{\tau}\int_{t-\tau}^{t} \left(\frac{\sigma(t')}{\sigma_Y}\right)^{\beta} dt' \leq 1$
Dynamic melting [7]	$G(t) = \sigma(t)$ – local stress, $G_C = \sigma_M$ – static melting limit, $\beta = 1$	$\frac{1}{\tau}\int_{t-\tau}^{t} \frac{\sigma(t')}{\sigma_M} dt' \leq 1$
Cavitation in fluids [5]	$G(t) = P(t)$ – local pressure, $G_C = P_C$ – mechanical strength of liquid, $\beta = 1/2 \div 1$	$\frac{1}{\tau}\int_{t-\tau}^{t} \frac{P_s^{\beta}(t')}{P_C^{\beta}} dt' \leq 1$
Electrical breakdown in insulators [5]	$G(t) = E(t)$ – the intensity of electric field, $G_C = E_C$ – electric strength, $\beta = 1$	$\frac{1}{\tau}\int_{t-\tau}^{t} \frac{P_s^{\beta}(t')}{P_C^{\beta}} dt' \leq 1$

Table 1. Dynamic transient processes obeyed the incubation time criterion

Let us note that every listed phenomenon demonstrates similar character of temporal dependence of strength having two (asymptotic) branches – the dynamic branch (determined by the value of incubation time τ) and the static one determined by the value G_C). It particularly means that the incubation time criterion (1) *degenerates into classical strength criterion* $G_s \leq G_C$ for long-term (quasistatic) fracture processes.

Structural-Continual Approach

So, a variety of different dynamic transient phenomena anticipated by relaxation-type processes obey the incubation time criterion that reveals the fundamental role played by incubation time regarding to abrupt structural changes in continuum. But incubation time criterion (1) allows an integral consideration of relaxation processes and does not provide their continual description at the microscale. And in 2008 *the* structural-*continual (temporal) approach* based on the notion of incubation time was presented [9] providing the kinetic description of abrupt structural changes in continuum. Proposed approach operates with *the damage function* which can be considered in fracture problems as instant local microfracture state (this function describes the microfracture evolution anticipating the macrofracture event including the processes of nucleation, interaction and following coalescence of microfracture – microcracks, microdamage and so on).

Let us consider a *spatially isotropic* process of microfracture evolution and fix an arbitrary small solid volume. Its mass is denoted as m, its volume before deformation is V_0, whereas the total volume of microfracture (damage) accumulated inside the chosen portion is V_*. Thus, during the damage accumulation process its volume changes as $V = V_0 + V_*$. The change of volume is obviously accompanied by variation of local density, described by the mass conservation law (or the charge conservation law for electrical breakdown phenomenon). Introducing the damage function as $\theta = dV_*/dV$, under the most common assumptions on the form of expression for the divergence of local velocity of medium particles [4], the mass conservation law takes on form

$$\frac{d\theta}{dt} = \frac{1}{\tau\zeta}\frac{G_s^\beta(t) - G_s^\beta(t-\tau)}{G_C^\beta}\theta^{\alpha_1}(1-\theta)^{\alpha_2}. \quad (2)$$

The way of definition of the damage function θ defines the range of it values: $\theta \in (-\infty, 1]$. Meanwhile $\theta = 0$ corresponds to intact (defectless) material, the local state of macroscopic failure is referred to $\theta = 1$ whereas $\theta < 0$ can be treated as some "suppressed" state. Therefore, the initial condition for Eq. 2 and the criterion of macro-failure obviously could be stated as

$$\theta(0) = 0\,, G(0) = 0 \text{ supposing } G(t \to 0+) \neq 0 \quad \text{and} \quad \theta(t_*) = 1, \quad (3)$$

where t_* is the time to fracture (time from the moment of loading application till the moment of macroscopic fracture). Being supplied with conditions (3) the equation (2) *generalizes incubation time approach* degenerating into (1) in the moment of macrofracture $t = t_*$. Unknown dimensionless parameters ζ, α_1 and α_2 in Eq. 2 have to be defined *by satisfying to known fracture criteria in particular cases*.

Definition of the parameters. Long-term quasistatic fracture

Thus, from correspondence of fracture criterion (3) to incubation time criterion (1) one could obtain [4]

$$\zeta = \frac{\Gamma(2-\alpha_1-\alpha_2)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}, \qquad 0 \le \alpha_1, \alpha_2 < 1. \quad (4)$$

Here Γ denotes Euler Gamma-function. Both parameters α_1, α_2 and β could be determined from analysis of long-term quasistatic fracture processes (when incubation time τ is neglected in comparison with the time to fracture t_*, that is $\tau \to 0$ could be supposed).

Creepage. For example, let consider the quasi-static fracture of initially "defectless" material – creepage problem (the tension of uniform bar by constant external load P – the specific load per the unit of initial cross-sectional area). Then, Eq. 2 yields [4]

$$\frac{d\theta}{dt} = \frac{\beta}{\zeta\sigma_C^\beta}P^\beta(1-\theta)^{-\zeta-\beta}, \quad \alpha_1 = 0, \alpha_2 = 1-\zeta. \quad (5)$$

It coincides exactly with Rabotnov creepage equation [10]

$$\frac{d\theta}{dt} = BP^k(1-\theta)^{-r-k} \quad \text{if } B = \frac{1}{\zeta\sigma_C^\beta}, \quad \beta = k, \quad \zeta = r. \quad (6)$$

So, structural-continual approach (2-3) generalizes fracture criteria both in static and dynamic cases as well as Rabotnov equations (for creepage). The dependences (6) could be easily verified by experimental data. In [11] the results of creep experiments made on under different loadings (40, 50, 60 и 80 MPa) are reported and the following parameters of Rabotnov equation (6) are calculated according to 84 experimental points: $B = 9.67 \cdot 10^{-10}$ $\text{MPa}^{-3}/\text{sec}$, $r = 2.38$, $k = 3.17$. From the other hand, for steel 12X18H10T (having a strength limit $\sigma_C = 530$ MPa) the dependence (6) gives $B = 9.72 \cdot 10^{-10}$ $\text{MPa}^{-3}/\text{sec}$ (that is the error is less than 5%).

Fatigue. Also, structural-continual approach parameters could be determined from another widespread quasistatic experiments. Applying Eq. 2 to analysis of fatigue crack growth under external cyclic loading $\sigma_0(t) = \frac{\Delta\sigma}{2}\left(1 + \sin\left(\frac{2\pi}{T}t - \frac{\pi}{2}\right)\right)$ having a period T and an amplitude $\Delta\sigma$ (see Fig. 1), one could obtain the dependence of crack growth rate on the stress intensity range ΔK_I in the form [4]

$$\frac{\Delta a}{\Delta N} = \frac{2}{\pi}\left(\frac{K_{IC}}{\sigma_C}\right)^2\left(\frac{\Delta K_I}{K_{IC}}\right)^\beta, \quad \alpha_1 = 1 - \zeta, \alpha_2 = 0. \tag{7}$$

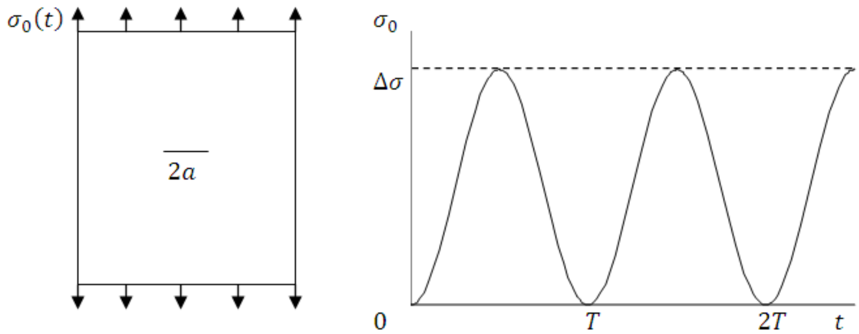

Fig. 1. Fatigue test arrangement

Comparison of Eq. 7 with Paris equation [12]

$$\frac{\Delta a}{\Delta N} = C_p(\Delta K_I)^{m_p}, \tag{8}$$

commonly used for description of fatigue crack propagation, yields

$$C_p = \frac{2}{\pi}\frac{K_{IC}^{2-\beta}}{\sigma_C^2} \quad \text{and} \quad m_p = \beta. \tag{9}$$

Table 2 contains the fatigue strength parameters and the right part of Eq. 9 calculated on over than 120 experimental points for different loading amplitudes (from 50 up to 150 MPa) obtained in standard fatigue tests (for plates 150x70x2 mm containing central crack 3.4 cm) for aircraft aluminum alloys 2024-T3 and 7075-T6 [13-14] (the error of dependence (9) is less than 9%).

	σ_C	K_{IC}	m_p	C_p	$\frac{2}{\pi}\frac{K_{IC}^{2-m_p}}{\sigma_C^2}$
2024 – T3	483 MPa	102.2 MPa$\sqrt{\text{m}}$	2.76	$8.61 \cdot 10^{-8}$	$8.24 \cdot 10^{-8}$
7075 – T6	586 MPa	74.1 MPa$\sqrt{\text{m}}$	3.08	$1.84 \cdot 10^{-8}$	$1.81 \cdot 10^{-8}$

Table 2. Fatigue strength characteristics of aluminum alloys 2024-T3 and 7075-T6

By the way, we have obtained the remarkable result in damage and fatigue mechanics – *the analytical relations between the parameters of Rabotnov and Paris equations*. Indeed, Eqs. (6) and (9) yield

$$B = \frac{1}{\zeta \sigma_C^{\beta}}, \quad C_p = \frac{2}{\pi} \frac{K_{IC}^{2-\beta}}{\sigma_C^2}, \quad \beta = m_p = k, \quad \zeta = r \quad \Rightarrow \quad C_p = \frac{2}{\pi} K_{IC}^{2-k} (rB)^{2/k}. \tag{10}$$

That is, we have shown that phenomena of static, long-term and dynamic strength are described from the common positions by Eq. 2. And even more: the whole strength behavior of material under any loading conditions could be predicted having results of static tests (determined static material strength σ_C and fracture toughness K_{IC}) and any of creep, fatigue or dynamic tests.

Dynamic cycling loading

Exploitation of structural-continuum approach in classical dynamic fracture problems, namely the cleavage problem and the problem of dynamic crack initiation, one can find in [4]. In this paper the results concerning *the propagation of a crack under intensively changing dynamic cycling loading* ("dynamic fatigue") are reported.

As a preliminary, let consider a quasibrittle macrocrack *under dynamic mode I* loading when $G \equiv K_1$, $\alpha_2 = 0$, $\alpha_1 = 1 - \zeta$ [15] and Eq. 2 yields

$$\frac{d\theta}{dt} = \frac{1}{\tau\zeta} \frac{K_1^{\beta}(t) - K_1^{\beta}(t-\tau)}{K_{1C}^{\beta}} \theta^{1-\zeta}. \tag{11}$$

Denoting $\eta = \theta^{\zeta}$ and partially integrating Eq. 11 taking into account the conditions (3) one obtain:

$$\eta(t) = \frac{1}{\tau} \int_{t-\tau}^{t} \left(\frac{K_1(t')}{K_{1C}} \right)^{\beta} dt', \quad \eta(0) = 0, \quad K_1(0) = 0, \quad K_1(t \to 0+) \neq 0 \tag{12}$$

and the criterion of macrofracture as

$$\eta(t_*) = 1. \tag{13}$$

Now we could investigate the problem of microdamage accumulation in the tip of a *macrocrack under a series of before-threshold loading pulses* (when every single pulse is insufficient to cause macrofracture in correspondence with criterion (1)). Namely, we will consider a thin plate with central crack $2a$ under external lateral cyclic load (for the first cycle)

$$\sigma_0(t) = \begin{cases} \frac{\Delta\sigma}{2}\left(1 + \sin\left(\frac{4\pi}{T} t - \frac{\pi}{2}\right)\right), & 0 \le t \le \frac{T}{2} \\ 0, & \frac{T}{2} \le t \le T \end{cases}, \qquad T \ll \tau. \tag{14}$$

The shape of stresses (14) is shown at Fig. 3.

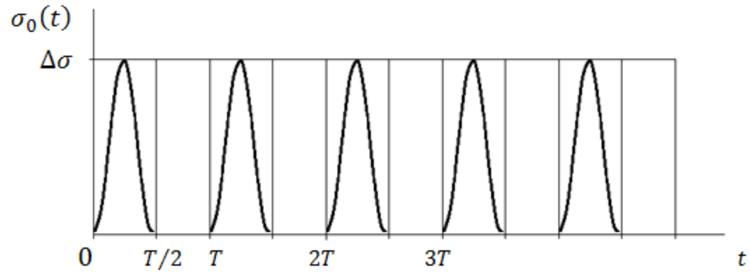

Fig 2. Temporal shape of external loading

Then the stress intensity factor till the moment of macrofracture (*crack advance for the length* d, see Eq. 1) is $K_1(t) = \sigma_0(t)\sqrt{\pi a}$. Substituting expressions (14) and (13) into Eq. (12) yields

$$\eta((k+1)T) - \eta(kT) = \beta_f \left(\frac{\Delta\sigma\sqrt{\pi a}}{K_{1C}}\right)^{\beta} \frac{T}{\tau}, \quad \beta_f = \frac{\sqrt{\pi}}{2} \frac{\sec(\pi\beta)}{\Gamma(1/2-\beta)\Gamma(1+\beta)}. \tag{15}$$

Here Γ denotes Euler Gamma-function. For most constructional materials (characterized by $1 \le \beta \le 6$) the parameter β_f is ranged by $0.1 \le \beta_f \le 0.25$ (upper limit corresponds to brittle materials with $\beta = 1$ [3]). Now, the number of load cycles causing macrofracture event (crack advance for the length d, see Eq. 1) under specified loading amplitude $\Delta\sigma$ could be calculated from Eqs. 13 and 15:

$$n_0 = \frac{1}{\beta_f}\left(\frac{K_{1C}}{\Delta\sigma\sqrt{\pi a}}\right)^{\beta} \frac{\tau}{T}. \tag{16}$$

Inverse problem (determination of minimal loading amplitude causing macrofracture at specified number of cycles) also could be solved. Minimal loading amplitude causing macrofracture $\Delta\sigma_{min}$ should correspond to the loading scheme implying the coincidence of the time to fracture and the incubation time $t_* = \tau$ that yields $T = \tau/n_0$, and then

$$\Delta\sigma_{min} = \beta_f^{-1/\beta} \frac{K_{1C}}{\sqrt{\pi a}}. \tag{17}$$

So, we have managed to describe the process of microdamage accumulation in the tip of a macrocrack under a series of before-threshold loading pulses and to determine the moment of increment of its half-length up to $a + d$.

Eq. 16 could be directly utilized to describe the process of *further crack propagation*. Indeed, at the consequent stages (for the crack having a half-length $a + kd$) the previous considerations remain valid and

$$n_{kd} = \frac{1}{\beta_f}\left(\frac{K_{1C}}{\Delta\sigma\sqrt{\pi(a+kd)}}\right)^{\beta} \frac{\tau}{T}, \quad k = 0,1,\ldots. \tag{18}$$

Therefore, the dependence between the growth rate of “dynamic fatigue” crack and the amplitude of external loading will have a form (*analogous to Paris equation* (8-9) for quasistatic case):

$$\frac{\Delta a}{\Delta n} = C_d(\Delta K_I)^{\beta}, \quad \text{where } C_d = \frac{2}{\pi}\frac{K_{IC}^{2-\beta}}{\sigma_C^2}\frac{T}{\tau}\beta_f, \quad \Delta K_1 = \Delta\sigma\sqrt{\pi a}. \tag{19}$$

And again the Eq. 19 involves just material constants measurable in quasi-static experiments and it has not only descriptive but also predictive power of complicated dynamic behavior of materials.

At Figs. 3 and 4 the results of numerical modeling of the processes of microdamage accumulation in the tip of a crack under dynamic pulsed loading and "dynamic fatigue" crack growth are plotted for standard fatigue samples (plates 150x70x2 mm containing a central crack 3.4 cm) of aircraft aluminum alloy 2024-T3.

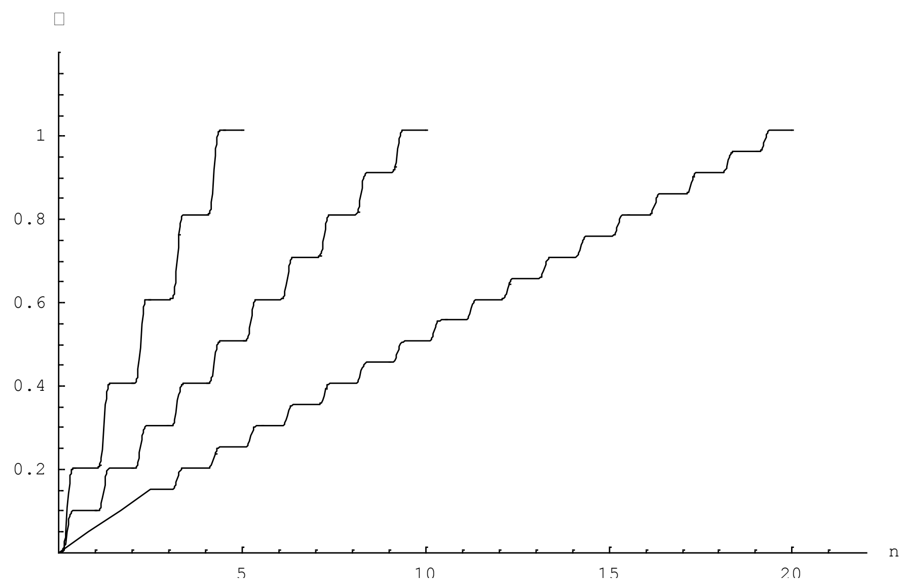

Fig. 3. Microdamage accumulation $\eta(n)$ under dynamic pulsed loading in 2024-T3 for different load cycling periods (from left to right: $T = 2\mu s,\ T = 4\mu s,\ T = 8\mu s$)

Stairs of constant damage function at Fig. 3 correspond to second-half fractions of loading periods (when loading is absent).

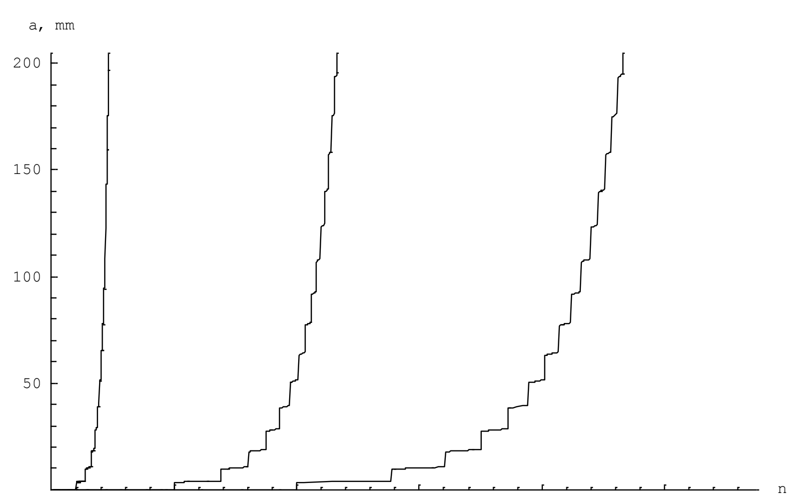

Fig. 4. Crack increment $a(n)$ under dynamic pulsed loading in 2024-T3 for different load cycling periods (from left to right: $T = 4\mu s,\ T = 0.2\mu s,\ T = 0.4\mu s$)

Summary
Revealing the fundamental role played by microfracture (relaxation) processes anticipating the dynamic macrofracture event has motivated the incubation time approach in dynamic fracture theory. But incubation time criterion allows an integral consideration of relaxation processes and does not provide a continual description of fracture evolution and corresponding incubation processes at the microscale. Further development of incubation time concept was made in the frameworks of structural-continual approach providing kinetic description of dynamic fracture processes. It operates with a function corresponding to instant local microfracture state (the damage function) to describe the microfracture evolution (including the processes of nucleation, interaction and following coalescence of microfracture – microcracks, microdamage and so on) anticipating the macrofracture event. Being generalization of incubation time criterion the correspondent kinetic equation involves just material constants measurable in quasi-static experiments and it has not only descriptive but also predictive power of complicated dynamic behavior of materials.
Exploitation of structural-continuum approach is illustrated on a range of quasi-static and dynamic fracture problems. By the way, the remarkable result in damage and fatigue mechanics is obtained – the analytical relations between the parameters of Rabotnov and Paris equations. It allows to predict the whole strength behavior of material under any loading conditions having results of static tests (determined static material strength σ_c and fracture toughness K_{Ic}) and any of creep, fatigue or dynamic tests. The abilities of proposed approach in description of the propagation of a crack under intensively changing dynamic cycling loading ("dynamic fatigue") are also discussed.
The main results can be easily generalized to describe sudden structural transitions in other problems of solid mechanics: pulsed electrical breakdown in insulators, cavitation in liquids, initiation of yielding, melting under high-rate loading, etc.

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