Shakedown of thin disks subject to thermomechanical loading

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Abstract

It is known that shakedown analysis is the extension of the plastic limit analysis for the structures under variable loads, and provides the safer criterion. A shakedown reduced kinematic formulation has been constructed for applications, which implies the plastic limit one as a limitting case. The present paper concerns, in particular, with thin elastic - plastic hollow disks under plane stress conditions subject to thermomechanical loading. The system of loading consists of a variable uniform pressure applied over the inner radius of the disk and a uniform temperature field varying with the time, while the outer radius of the disk is fixed. Due to the system of loading applied and the constraints imposed the problem is axisymmetric. Semianalytrical application of the reduced shakedown kinematic theorem produces two possible incremental and alternating plasticity collapse modes. Another plastic limit mode is when the plastic deformation occurs over entire disk. The collapse curve in parametric load-temperature space for the disk is taken as the lower envelope of those modes.

1. INTRODUCTION

It is known that the application of computational models to plane stress problems leads to specific difficulties non-existent in other formulations [1]. In particular, solutions may not exist for a certain set of input parameters or may be singular. Analytic solutions are very useful for studying such features of the model. Also, analytic solutions should be used to verify numerical codes. Determination of plastic limit states for thin plates and disks subjected to various kinds of loads has been pursued in many works, see e.g. [2-5]. In those studies one-parameter loading has been considered. Some problems related to the onset of plastic limit state of structures in the case of multi-parameter loading was studied in [6]. Thermal loading of thin disks under various kinds of constraints was examined in [7-11]. Various rigid-plastic and elasto-plastic solutions for thin plates with cylindrical hole have been provided in [12-13]. In [14] a semi-analytical solution has been constructed for plastic limit state of hollow disk under combined actions of temperature field and pressure on the internal contour (the external contour of the disk is presumed to be fixed). Various possible collapse modes have been analyzed. The result obtained would be used to treat a possible incremental collapse mode as we treat the more complicated shakedown problem for the structure in this work. Shakedown analysis is the extension of the plastic limit analysis for the structures under variable loads, and provides the safer criterion [15-19]. In [17-19] we produced a shakedown reduced kinematic formulation, which has been used successfully to solve various practical shakedown problem. In this work the semi-analytical approach would be applied to solve the problem for the disks, in particular, those examined in [14].

2. SHAKEDOWN AND PLASTIC LIMITS OF STRUCTURES

An elastic-plastic structure under given loading history, after a finite amount of initial plastic deformation may eventually shake down to some state, from which it subsequently responds elastically to the external agencies. Otherwise, the structure is considered as having failed, because of continuing plastic deformation.

Let $\boldsymbol{\sigma}^e(\boldsymbol{x},t)$ denote the fictitious stress response of the body V to external agencies over a period of time $(\boldsymbol{x} \in V, t \in [0,T])$ under the assumption of perfectly elastic behaviour. The actions of all kinds of external agencies upon V can be expressed explicitly through $\boldsymbol{\sigma}^e$. At every point $\boldsymbol{x} \in V$, the elastic stress response $\boldsymbol{\sigma}^e(\boldsymbol{x},t)$ is confined to a bounded time-independent local loading domain L_x. As a field over V, $\boldsymbol{\sigma}^e(\boldsymbol{x},t)$ belongs to the time-independent global loading domain L:

$$L = \left[\boldsymbol{\sigma}^e \mid \boldsymbol{\sigma}^e(\boldsymbol{x},t) \in L_x, \boldsymbol{x} \in V, t \in [0,T]\right] \tag{1}$$

In the spirit of classical shakedown analysis, the bounded loading domain L, instead of a particular loading history, is given a priori. Shakedown of a body in L means it shakes down for all possible loading histories $\boldsymbol{\sigma}^e(\boldsymbol{x},t) \in L$.

Let k_s denote the shakedown safety factor: at $k_s > 1$, the structure will shake down, while it will not at $k_s < 1$, and $k_s = 1$ defines the boundary of the shakedown domain. Following the kinematic upper bound approach of [5,6,7], we establish that:

$$k_s^{-1} = max\{I, A\} \tag{2}$$

where

$$I = \sup_{\boldsymbol{\sigma}^e \in L, \boldsymbol{\varepsilon}^p \in C} \frac{\int_V \max_{tx} \left[\boldsymbol{\sigma}^e(\boldsymbol{x},t_x) : \boldsymbol{\varepsilon}^p(\boldsymbol{x})\right] dV}{\int_V D(\boldsymbol{\varepsilon}^p) dV} \tag{3}$$

$$A = \sup_{\boldsymbol{\sigma}^e \in L; \boldsymbol{x} \in V; \boldsymbol{\varepsilon}^p; t; t'} \frac{\left[\boldsymbol{\sigma}^e(\boldsymbol{x},t) - \boldsymbol{\sigma}^e(\boldsymbol{x},t')\right] : \boldsymbol{\varepsilon}^p}{2D(\boldsymbol{\varepsilon}^p)} \tag{4}$$

where where C is the set of strain fields that are both deviatoric and compatible on V; $D(\varepsilon^p) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^p$ is the dissipation function determined by the yield stress σ_Y and the respective yield criterion, in particular, for a Mises material we have:

$$D(\boldsymbol{\varepsilon}^p) = \frac{2}{\sqrt{3}} \sigma_Y \left(\boldsymbol{\varepsilon}^p : \boldsymbol{\varepsilon}^p\right)^{1/2} \tag{5}$$

Not that the deviatoric plastic strain $\boldsymbol{\varepsilon}^p$ in (4), in contrary to that in (3), is not required to be a compatible field on $\boldsymbol{V}$. The compatible field $\boldsymbol{\varepsilon}^p$ in (3) represents the incremental plastic strain increment over a cycle. I represents the incremental collapse mode, and A- the alternating plasticity collapse one.

Through the plastic limit safety factor k_p, the plastic limit kinematic theorem can also be stated as:

$$k_p^{-1} = \sup_{\varepsilon^p \in C} \frac{\int_V \left[\boldsymbol{\sigma}^e(\boldsymbol{x}) : \boldsymbol{\varepsilon}^p(\boldsymbol{x})\right] dV}{\int_V D\left(\boldsymbol{\varepsilon}^p\right) dV} \tag{6}$$

which can be considered as a limiting case of the shakedown analysis (1)- (4) [3]. In (6), $\boldsymbol{\sigma}^e(\boldsymbol{x})$ is the fictitious elastic field, or any equilibrated stress field at collapse; $\boldsymbol{\varepsilon}^p$ is the plastic strain rate field at collapse. For quasistatic loading processes, one always has $I \ge k_p^{-1}$, hence $k_s^{-1} \ge k_p^{-1}$ and $k_s \le k_p$, which indicates that the shakedown criterion is safer than the plastic limit one for the processes.

3. SHAKEDOWN LIMIT FOR CIRCULAR DISKS

Consider an axis-symmetric problem for thin circular hollow disk of internal and external radii r_0 and R_0, respectively, subject to some external mechanical and thermal fields, varying within certain limits. The elastic stress field has three principal components σ_r^e, σ_θ^e, and $\sigma_z^e = 0$, in the cylindrical system of coordinates $\{r, \theta, z\}$. The plastic strain (rate) field also has three principal components ε_r^e, ε_θ^e, and ε_z^e, where $\varepsilon_z^p = -\varepsilon_r^p - \varepsilon_\theta^p$ from the plastic incompressibility. The shakedown formulation for that plane stress problem can be made into the particular form:

$$k_s^{-1} = max\{I, A\} \tag{7}$$

where

$$I = \sup_{\sigma^e \in L, \varepsilon^p \in C} \frac{\int_V \max_{t_r} \left[\sigma_r^e(r,t_r)\varepsilon_r^p(r) + \sigma_\theta^e(r,t_r)\varepsilon_\theta^p(r)\right] r dr}{\frac{2}{\sqrt{3}}\sigma_Y \int_{r_0}^{R_0} \left\{\left[\varepsilon_r^p(r)\right]^2 + \left[\varepsilon_\theta^p(r)\right]^2 + \varepsilon_r^p(r).\varepsilon_\theta^p(r)\right\}^{1/2}_{t_r} r\, dr} \tag{8}$$

$$\begin{aligned} A = \max_{r,t,t'} \frac{1}{2\sigma_Y} \Big\{ & \left[\sigma_r^e(r,t) - \sigma_r^e(r,t')\right]^2 - \left[\sigma_\theta^e(r,t) - \sigma_\theta^e(r,t')\right]^2 \\ & - \left[\sigma_r^e(r,t) - \sigma_r^e(r,t')\right]\left[\sigma_\theta^e(r,t) - \sigma_\theta^e(r,t')\right] \Big\}^{1/2} \end{aligned} \tag{9}$$

Presume the disk is under variable quasistatic mechanical and thermal fields. In particular, the disk is subjected to variable pressure $0 \le P \le P^+$ on the internal contour ($r = r_0$), in variable a homogeneous temperature increment $0 \le T \le T^+$ (from the zero reference environment temperature, at which the unloaded disk is free of stresses). The external contour of the disk ($r = R_0$) is fixed (see Fig.1). Introduce the dimensionless variables:

$$p = P / \sigma_Y,\ \rho = r / R_0,\ \rho_0 = r_0 / R_0,\ \tau = \frac{\alpha E T}{\sigma_Y}, \tag{10}$$

where α and E are the thermal expansion coefficient and elastic Young modulus. The thermoelastic stress solution for the disk is:

$$\bar{\sigma}^e_\rho = \sigma^e_r / \sigma_Y = a/\rho^2 + b, \qquad \bar{\sigma}^e_\theta = \sigma^e_\theta / \sigma_Y = -a/\rho^2 + b$$

$$a = \frac{\rho_0^2 \left[\tau - (1-\nu)p\right]}{\rho_0^2(1+\nu) + 1 - \nu}, \quad b = \frac{(1+\nu)\rho_0^2 p + \tau}{(1+\nu)\rho_0^2 + 1 - \nu} \tag{11}$$

$$\rho_0 \le \rho \le 1, 0 \le p \le p^+, 0 \le \tau \le \tau^+$$

where ν is Poinsson ratio.

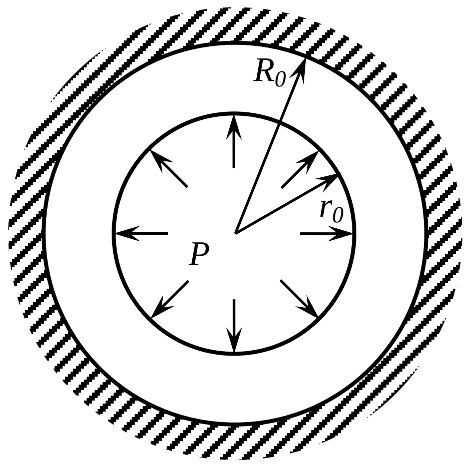

Fig.1. A circular disk

In shakedown analysis, (11) is used to represent the actions of external agencies over the disk in the dimensionless expressions of the incremental and alternating plasticity collapse modes:

$$I = \sup_{\sigma^e \in L, \varepsilon^p \in C} \frac{\int_{\rho_0}^{1} \max_{t_\rho} \left[\bar{\sigma}^e_\rho(\rho, t_\rho)\varepsilon^e_\rho(\rho) + \bar{\sigma}^e_\theta(\rho, t_\rho)\varepsilon^e_\theta(\rho)\right]\rho\, d\rho}{\frac{2}{\sqrt{3}} \int_{\rho_0}^{1} \left\{\left[\varepsilon^p_\rho(\rho)\right]^2 + \left[\varepsilon^p_\theta(\rho)\right]^2 + \varepsilon^p_\rho(\rho)\varepsilon^p_\theta(\rho)\right\}^{1/2} \rho\, d\rho} \tag{12}$$

$$A = \max_{\rho,t,t'} \frac{1}{2}\left\{\left[\bar{\sigma}^e_\rho(\rho,t) - \bar{\sigma}^e_\rho(\rho,t')\right]^2 + \left[\bar{\sigma}^e_\theta(\rho,t) - \bar{\sigma}^e_\theta(\rho,t')\right]^2 - \left[\bar{\sigma}^e_\rho(\rho,t) - \bar{\sigma}^e_\rho(\rho,t')\right]\left[\bar{\sigma}^e_\theta(\rho,t) - \bar{\sigma}^e_\theta(\rho,t')\right]\right\}^{1/2} \tag{13}$$

Equation $I = 1$ defines the incremental collapse mode in the loading space $\{\tau, p\}$, while $A = 1$- the incremental collapse mode. The lower envelope of them defines the shakedown boundary $k_s = 1$, under which the disk is safe.

In the mean time, the respective plastic limit of the disk is determined by:

$$k_p^{-1} = \sup_{\varepsilon^p \in C} \frac{\int_{\rho_0}^{1} \left[\bar{\sigma}^e_\rho(\rho)\varepsilon^p_\rho(\rho) + \bar{\sigma}^e_\theta(\rho)\varepsilon^p_\theta(\rho)\right]\rho\, d\rho}{2\sqrt{2/3} \int_{\rho_0}^{1} \left\{\left[\varepsilon^p_\rho(\rho)\right]^2 + \left[\varepsilon^p_\theta(\rho)\right]^2 + \left[\varepsilon^p_\rho(\rho)\right]\left[\varepsilon^p_\theta(\rho)\right]\right\}^{1/2} \rho\, d\rho} \tag{14}$$

Equation $k_p = 1$ defines the plastic limit boundary in the loading space $\{\tau, p\}$, under which the disk is safe.

An analytical solution for the instant of plastic yielding over the whole of the disk has been offered in [14]. The total kinematic field of the problem at the instant has been found as:

$$\varepsilon_\theta = \lambda\left\{1 - \exp\left[\sqrt{3}(\gamma_m - \gamma)\right]\right\}$$

$$\varepsilon_z = \lambda\left\{1 + \frac{1}{2}\exp\left[\sqrt{3}(\gamma_m - \gamma)\right]\left(1 + \sqrt{3}\tan\gamma\right)\right\}, \qquad (15)$$

where λ is a proportional coefficient, γ depends on ρ as:

$$\frac{\rho}{\rho_0} = \left(\frac{\sqrt{3}\cos\gamma_0 - \sin\gamma_0}{\sqrt{3}\cos\gamma - \sin\gamma}\right)^{1/2} exp\left[\frac{\sqrt{3}}{2}(\gamma - \gamma_0)\right] \qquad (16)$$

γ_0 and γ_m can be found from:

$$\sin\gamma_0 = \frac{\sqrt{3}}{2}p, \quad \sqrt{3}\cos\gamma_m + (1 - 2\nu)\sin\gamma_m = \sqrt{3}\tau \qquad (17)$$

Solution for the alternating plasticity collapse mode (13) can be found directly, using (11). Problems (12) and (14) are nonlinear optimization problems, and in the general case should be solved numerically. With appropriate trial kinematic collapse fields, one may find from them (upper bound) estimates for the respective collapse loads. A possible incremental collapse mode and plastic limit one can be found by substituting a localized compatible plastic strain trial field ($\delta(\rho)$ is Dirac delta function):

$$\varepsilon_r^p = -\varepsilon_z^p = \delta(\rho - c), \ \rho_0 \le c \le 1 \qquad (18)$$

into (12) and (14), and optimize them over the variable c (called the collapse mode I). The alternating plasticity collapse mode is found directly from (13). The plastic limit state (15)-(17) is called here the collapse mode II.

For numerical illustrations, we take $\rho_0 = 0.5; \nu = 0.3$. In Fig. 2, the incremental collapse and plastic limit curve $I = k_p^{-1} = 1$ using the trial field (18) (mode I- solid line), together with the plastic limit curve from [14] (mode II - doted line), are projected in Fig. 2. The alternating plasticity collapse curve $A = 1$ is also given for three cases:

(a) $0 \le p \le p^+, \ 0 \le \tau \le \tau^+;$

(b) $0.1p^+ \le p \le p^+, \ 0 \le \tau \le \tau^+;$

(c) $0 \le p \le p^+, \ 0.1\tau^{\div} \le \tau \le \tau^+$

Shakedown safety domain should lie under all those curves: The collapse modes I and II, and one of the alternating plasticity collapse curves (a),(b), or (c), in the respective cases.

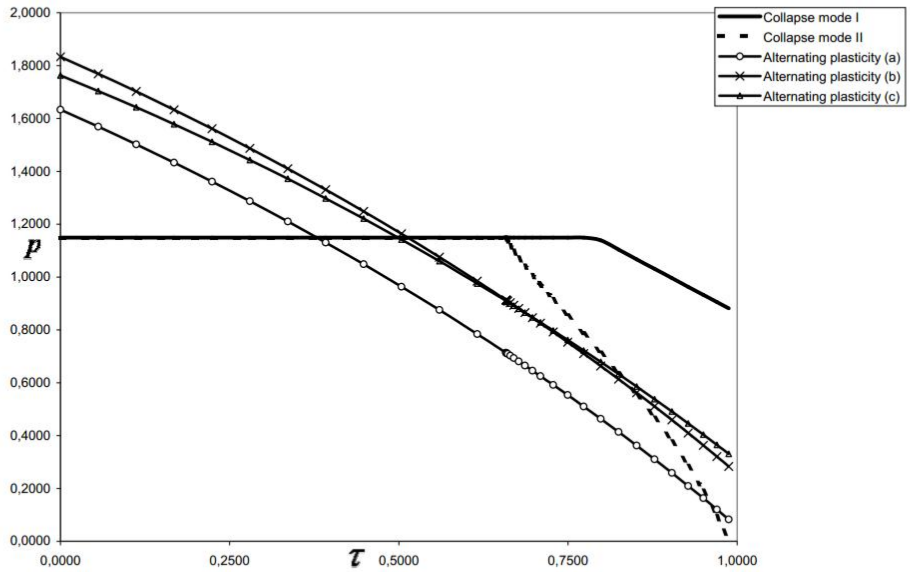

Fig.2. Collapse curves

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