On the failure of anisotropic materials: Physical or Mathematical?

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Abstract. In order to understand the mechanism of failure (yielding or fracture) of linear elastic anisotropic materials we examine the qualitative characteristics of two widely used methods in fracture mechanics:

a. The physical decomposition of the elastic strain energy density into a dilatational (volume changes) and a distortional (shape changes) one.
b. The mathematical (spectral) decomposition of the compliance tensor which leads to up to six independent energy-orthogonal terms.

Whether one uses the first or the second proposed decomposition, this should obey to an undeniable physical truth: "none of the energetic terms can rise to infinity". By precisely measuring those limits one can easily extract the failure box for a given material (or the safe states if preferred). If now, both of the above used methods are of some meaning for the failure of the material, then both of the extracted failure boxes should limit the same or almost the same area in the principal stress space. Results show that the two failure boxes differ significantly in a qualitative and a quantitative manner. A discussion on these results is presented.

Introduction

In the present analysis, we refer to linear elastic materials. Therefore, the constitutive equation they obey in will be the Generalized Hooke's Law

$$\underset{\sim}{\sigma} = \underset{\sim}{C} \cdot \underset{\sim}{\varepsilon} \text{ or } \underset{\sim}{\varepsilon} = \underset{\sim}{S} \cdot \underset{\sim}{\sigma} \tag{1}$$

where $\underset{\sim}{C}$ is the stiffness tensor and $\underset{\sim}{S}$ is the compliance tensor. The stress and strain tensors are represented as vectors in a 6-dimensional space, according to Voigt's notation. So,

$$\underset{\sim}{\sigma} = \left[\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4, \ \sigma_5, \ \sigma_6\right]^{\mathrm{T}} \text{ and } \underset{\sim}{\varepsilon} = \left[\varepsilon_1, \ \varepsilon_2, \ \varepsilon_3, \ \varepsilon_4, \ \varepsilon_5, \ \varepsilon_6\right]^{\mathrm{T}}. \tag{2}$$

The strain energy density of a linear elastic material can be defined as:

$$T = \frac{1}{2}\underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon} = \begin{cases} \frac{1}{2}\underset{\sim}{\sigma} \cdot \underset{\sim}{S} \cdot \underset{\sim}{\sigma} & \text{in stress terms} \\ \frac{1}{2}\underset{\sim}{\varepsilon} \cdot \underset{\sim}{C} \cdot \underset{\sim}{\varepsilon} & \text{in strain terms} \end{cases} \tag{3}$$

Physical Decomposition. The first method, a physical one, is used by the authors with great success in the case of isotropic linear and non-linear elastic materials [1] in order to predict failure. The compliance tensor of an isotropic material is of the form:

$$\underset{\sim}{S} = \begin{bmatrix} s_{11} & s_{12} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{12} & 0 & 0 & 0 \\ s_{12} & s_{12} & s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(s_{11}-s_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{bmatrix} \quad (\textit{2 independent parameters}) \quad (4)$$

With respect to the above imposed symmetries, the quality of the stress tensor that causes volume changes (equal normal strains $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = m$) is that of a hydrostatic stress tensor (equal normal stresses $\sigma_1 = \sigma_2 = \sigma_3 = p$). Given a stress tensor $\underset{\sim}{\sigma}$, one can extract a hydrostatic part $\underset{\sim}{\sigma}_p$, where

$$\underset{\sim}{\sigma}_p = (\sigma_1 + \sigma_2 + \sigma_3)\left[\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ 0,\ 0,\ 0\right]^{\mathrm{T}} \quad (5)$$

and a deviatoric one $\underset{\sim}{\sigma}_d$, where

$$\underset{\sim}{\sigma}_d = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}_p = \left[\frac{2\sigma_1 - \sigma_2 - \sigma_3}{3},\ \frac{-\sigma_1 + 2\sigma_2 - \sigma_3}{3},\ \frac{-\sigma_1 - \sigma_2 + 2\sigma_3}{3},\ \sigma_4,\ \sigma_5,\ \sigma_6\right]^{\mathrm{T}} \quad (6)$$

Now, this hydrostatic tensor $\underset{\sim}{\sigma}_p$ is the unique one (from the infinite number of possible forms) that is responsible for the whole volume change of the material. Consequently, the deviatoric stress tensor $\underset{\sim}{\sigma}_d$ is responsible for the whole shape change of the material. In conclusion, we have

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_p + \underset{\sim}{\sigma}_d \quad (7)$$

and

$$T = \frac{1}{2}\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon} = \frac{1}{2}\underset{\sim}{\sigma}\cdot\underset{\sim}{S}\cdot\underset{\sim}{\sigma} = \frac{1}{2}\left(\underset{\sim}{\sigma}_p + \underset{\sim}{\sigma}_d\right)\cdot\underset{\sim}{S}\cdot\left(\underset{\sim}{\sigma}_p + \underset{\sim}{\sigma}_d\right) = \frac{1}{2}\underset{\sim}{\sigma}_p\cdot\underset{\sim}{S}\cdot\underset{\sim}{\sigma}_p + \frac{1}{2}\underset{\sim}{\sigma}_d\cdot\underset{\sim}{S}\cdot\underset{\sim}{\sigma}_d \quad (8)$$

where

$T_V = \frac{1}{2}\underset{\sim}{\sigma}_p\cdot\underset{\sim}{S}\cdot\underset{\sim}{\sigma}_p$ is the energy associated with *volume changes only* and (9)

$T_D = \frac{1}{2}\underset{\sim}{\sigma}_d \cdot \underset{\sim}{S} \cdot \underset{\sim}{\sigma}_d$ is the energy associated with *shape changes only*. (10)

For the case of anisotropic materials though, one has to be very careful with the implementation of such decomposition. There is a high tendency to confuse volume changes with the application of the hydrostatic pressure only. But this is not correct in general. In fact, this is the main reason why all the previous attempts to decompose the elastic strain energy failed [2]. We present here a physical way of determining correctly the dilatational and the distortional energetic parts for a material possessing hexagonal symmetry. The same procedure can be followed for the rest of the anisotropic types of materials.

The compliance tensor of a material possessing hexagonal symmetry is of the form:

$$\underset{\sim}{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{bmatrix} \quad (5 \textit{ independent parameters}) \tag{11}$$

Now with respect to these symmetries, which are different from those of an isotropic material, the quality of the stress tensor that causes volume changes (equal normal strains $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = m$) is of the form

$$\underset{\sim}{\sigma}_p = \left[\frac{(-s_{13}+s_{33})m}{-2s_{13}^{\ 2}+(s_{11}+s_{12})s_{33}}, \frac{(-s_{13}+s_{33})m}{-2s_{13}^{\ 2}+(s_{11}+s_{12})s_{33}}, \frac{(s_{11}+s_{12}-2s_{13})m}{-2s_{13}^{\ 2}+(s_{11}+s_{12})s_{33}},\ 0,\ 0,\ 0 \right]^T, \tag{12}$$

clearly not a hydrostatic one! One cannot choose whichever value for the parameter m on the above formula, but only this one that leads the term $\underset{\sim}{\sigma}_p \cdot \underset{\sim}{S} \cdot \underset{\sim}{\sigma}_d$ in Eq.(8) to zeroing. For the hexagonal anisotropic materials this is equal to

$$m = \frac{(-2s_{13}^{\ 2}+(s_{11}+s_{12})s_{33})(\sigma_1+\sigma_2+\sigma_3)}{s_{11}+s_{12}-4s_{13}+2s_{33}}. \tag{13}$$

Therefore,

$$\underset{\sim}{\sigma}_p = (\sigma_1+\sigma_2+\sigma_3)\left[\frac{(s_{33}-s_{13})}{s_{11}+s_{12}-4s_{13}+2s_{33}}, \frac{(s_{33}-s_{13})}{s_{11}+s_{12}-4s_{13}+2s_{33}}, \frac{(s_{11}+s_{12}-2s_{13})}{s_{11}+s_{12}-4s_{13}+2s_{33}}, 0,0,0 \right]^T \tag{14}$$

Inserting the above value of $\underset{\sim}{\sigma}_p$ to the Eqs.(7)-(10) one is able to determine the strain energy T_V required for volume changes only and the strain energy T_D required for shape changes only.

Mathematical Decomposition. The second one is a pure mathematical method with an obscure and sometimes dark physical meaning. The diagonalization of the compliance tensor $\underset{\sim}{S}$ (with its eigenvalues λ_i as its diagonal terms) leads to six at most independent eigenstress tensors $\underset{\sim}{\sigma}_{mi}$. This way, the Generalized Hooke's law can be decomposed into 6 at most simple (uniaxial) Hooke's laws [3], through which the total elastic strain energy density can be decomposed into six independent energetic terms,

$$\underset{\sim}{\sigma}_{mi} = \lambda_i \cdot \underset{\sim}{\varepsilon}_{mi} \tag{15}$$

and

$$T = \frac{1}{2}\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon} = \frac{1}{2}\left(\sum_{i=1}^{6}\underset{\sim}{\sigma}_{mi}\right)\cdot\left(\sum_{i=1}^{6}\underset{\sim}{\varepsilon}_{mi}\right) = \frac{1}{2}\sum_{i=1}^{6}\left(\underset{\sim}{\sigma}_{mi}\cdot\underset{\sim}{\varepsilon}_{mi}\right) \tag{16}$$

where $T_i = \frac{1}{2}\underset{\sim}{\sigma}_{mi}\cdot\underset{\sim}{\varepsilon}_{mi}$, i=1,…,6 are the six independent energetic terms.

The compliance tensor $\underset{\sim}{S}$ of a material possessing hexagonal symmetry has five distinct eigenvalues. So, the spectral decomposition leads to five distinct non-interactive energetic terms.

Numerical example. Let us assume that we would like to investigate the qualitative characteristics of the failure box of an hexagonal anisotropic material, i.e. Beryllium. The elastic constants of Beryllium are shown in Table 1 below.

Table 1. Elastic constants of Beryllium [4]

c_{11} [GPa]	c_{33} [GPa]	c_{44} [GPa]	c_{12} [GPa]	c_{13} [GPa]
292.3	336.4	162.5	26.7	14

For simplicity reasons (in order to avoid plasticity effects), let as assume that failure occurs when the material reaches its yield point. We are going to plot the failure boxes (more specifically the yielding boxes) obtained by the physical and the mathematical decomposition of the strain energy density function. Both of the failure boxes were extracted by the notion that none of the energetic terms can keep rising up to infinity. We assumed that all of the energetic terms can rise up to the value of, say, 5 GPa. In fact, the distinct energetic terms have usually distinct energetic boundaries, but for a qualitative representation the same conclusions can be derived for a unique energetic boundary value. The results are shown in Figure 1.

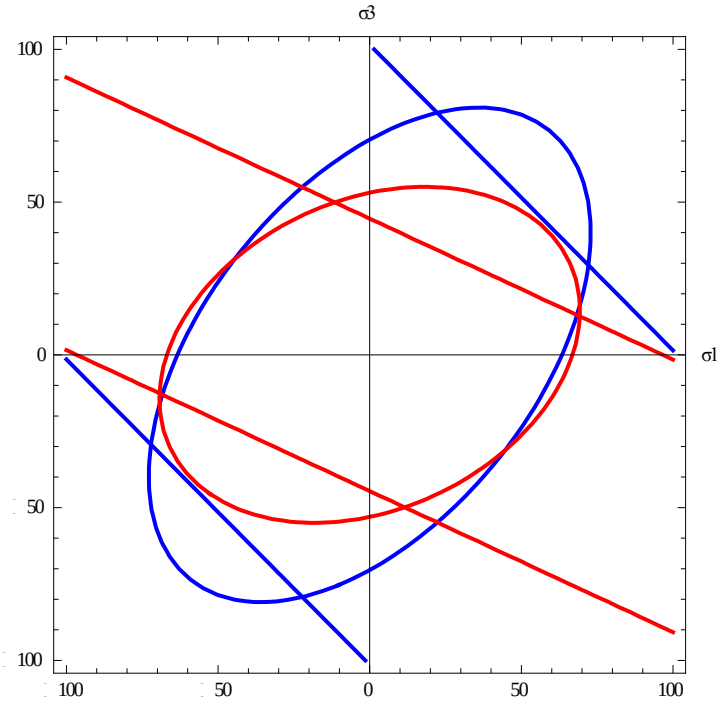

Fig.1. Physical (blue) and Mathematical Decomposition (red) Based Failure Boxes of Beryllium.

In the above figure the results are plotted in the $\sigma_1 - \sigma_3$ stress space because $\sigma_1 - \sigma_2$ is an isotropic plane. We have drawn with blue lines the failure box obtained by bounding the energetic terms T_V and T_D (corresponding to volume change and shape change respectively). The set of parallel blue lines corresponds to volume changes and the ellipsoidal part corresponds to shape changes (alike the Mises boundary).

Red lines correspond to three out of the five distinct energetic terms obtained by the spectral decomposition of the compliance tensors (Eq.16). Specifically, we have chosen arbitrarily the sum of T_i, $i = 3,4,5$ to obtain the ellipsoidal curve and T_4 alone for the set of parallel red lines.

The physical decomposition gives qualitatively similar results as in case of isotropy but nor the axis of ellipsoid $(T_D = \text{const.})$ neither the direction of the set of parallel lines $(T_V = \text{const.})$ are symmetric with respect to the diagonals of the space $\sigma_1 - \sigma_3$.

The mathematical decomposition results in a set of parallel lines for each distinct energetic term. A combination of more than one energetic term is required to obtain higher order curves and there is no rule for the selection of proper combinations to decide on what is the physical reaction of the material.

Summary

In the present work, two approaches are presented for the solution of the long pending problem of the separation of elastic strain energy density in case of linear anisotropic materials. Comparison of the results obtained indicates that the so-called physical decomposition is by far more reasonable and clear than those of the mathematical one. The main handicap of the latter method is that we cannot have sense on the effect on the material geometry of each of the six

energetic terms. In fact, it is impossible, in general, to predict if a certain energetic term T_i causes volume, shape or both changes in the material. It seems that physical considerations are, absolutely, necessary for the solution of the present problem.

It is hoped that the present physical decomposition may drive to better understanding of the failure behavior of materials

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