Nonlinear eigenvalue problems arising from nonlinear fracture mechanics analysis

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Abstract. The paper is devoted to the fatigue growing crack problems in damaged media and mutual effects of damage on the evolution of the stress-strain state near the crack tip and vice versa. The new asymptotic study of fatigue crack growth in an isotropic linear elastic material based on the continuum damage mechanics in the coupled (elasticity – damage) formulation under plane strain and plane stress conditions is realized. The numerical solution of the two-point boundary value problem for non-linear ordinary differential equations to which the fatigue crack growing problem reduces is obtained. The new analytical presentation of stress, strain and continuity fields both for plane strain and plane stress conditions is given. The results obtained differ from Zhao and Zhang's solution where the original formulation of the problem for plane stress conditions has been proposed. The analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is justified by the perturbation theory technique used. It is shown that the perturbation method allows to find the analytical formula expressing the eigenvalue as the function of parameters of the damage evolution law.

Introduction. Accurate description of crack-tip stress and deformation fields is the basis to establish a proper macroscopic fracture criterion and predict the failure of cracked structures. The interest in the current paper is to reveal the asymptotic crack-tip fields in power-law material and (or) in a damaged material with power law damage evolution equation. Nowadays fracture process is considered as a multiscale process. It is necessary to distinguish the fracture process at macro-, meso- and microscopic level and to include into consideration damage accumulation at different levels either. Developing multiscale fracture and damage models implies hierarchy of stress singularities: strong and weak singularities reflecting fracture and damage accumulation at different scales. Stress singularities can be expressed as r^{λ} with λ being the order of the singularity. To obtain reasonable description of fracture process an analytical solution $\lambda = \lambda(n,m)$ where n,m are material parameters of stress-strain relations and damage evolution law for all eigenvalues similar to the HRR stress field with theoretically known formula $\lambda = -1/(n+1)$ is needed.

Asymptotic study of fatigue crack growth in a damaged medium. Consider a fatigue growing crack lying on the x-axis with the coordinates origin located at the moving crack tip as shown in Fig. 1. The constitutive equations are formulated in the framework of continuum damage mechanics as

$$\varepsilon = \frac{1+\nu}{E}\frac{\sigma_{ij}}{\psi} - \frac{\nu}{E}\frac{\sigma_{kk}}{E}\delta_{ij}, \qquad (1)$$

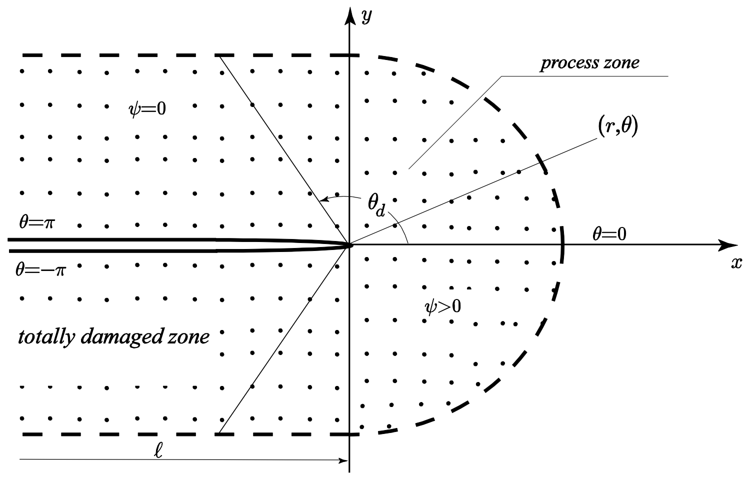

Fig.1. Geometry of a fatigue crack in a damaged medium

where E and ν are the Young's modulus and Poisson's ratio, respectively, and the continuity variable evolutes according to

$$\frac{d\psi}{dN}=\begin{cases}-c\left(\frac{\sigma_e}{\psi}\right)^m\frac{1}{\psi^{n-m}}, & \sigma_e\geq\sigma_{th}\psi^{\gamma}\\ 0, & \sigma_e\leq\sigma_{th}\psi^{\gamma}\end{cases} \tag{2}$$

where N is the number of cycles, c,n,m,γ and σ_{th} are material constants.

The original statement of the problem is proposed in [1]. The asymptotic solution of fatigue crack growth based on continuum damage mechanics given by Jin Zhao and Xing Zhang has aroused considerable interest and is presented in books [2, 3] in details. However the results obtained in [1] do not agree with the conclusions formulated in other works devoted to crack tip problems in materials with the damage coupled stress-strain relations [3, 4]. Recent interest in the development of multiscale fracture models has motivates extension of the asymptotic study of fatigue crack growth based on damage mechanics. Following [1] the problem of plane stress and plane strain for mode I is considered under small-scale damage conditions and the equilibrium equations are satisfied by introducing the Airy stress function

$$F(r,\theta)=\alpha r^{\lambda+2}f(\theta). \tag{3}$$

The stress components near a crack tip are separable and can be expressed as

$$\sigma_{rr}(r,\theta)=\alpha r^{\lambda}\tilde{\sigma}_{rr}(\theta),\ \sigma_{\theta\theta}(r,\theta)=\alpha r^{\lambda}\tilde{\sigma}_{\theta\theta}(\theta),\ \sigma_{r\theta}(r,\theta)=\alpha r^{\lambda}\tilde{\sigma}_{r\theta}(\theta),$$

where $\tilde{\sigma}_{rr}=(\lambda+2)f(\theta)+f''(\theta),\ \tilde{\sigma}_{\theta\theta}=(\lambda+2)(\lambda+1)f(\theta),\ \tilde{\sigma}_{r\theta}=-(\lambda+1)f'(\theta)$.

The continuity field around the crack tip is presented in the form

$$\psi(r,\theta)=\beta r^{\mu}g(\theta). \tag{4}$$

The compatibility equation, the kinetic law (2) and asymptotic expansions (3) and (4) result in the following nonlinear eigenvalue problem

$$f^{IV}-2\overline{E}f'''+\left(\overline{G}+b_1\right)f''-b_2\overline{E}f'+\left(b_3+e_1\overline{G}\right)f=0 \tag{5}$$

for plane stress conditions;

$$f^{IV}-2\overline{E}f'''+\left(\overline{G}+d_1\right)f''+d_2\overline{E}f'+\left(d_3+s_1\overline{G}/(1-\nu)\right)f=0 \tag{6}$$

for plane strain conditions;

$$g'\sin\theta-\mu g\cos\theta=-\tilde{\sigma}_e^m g^{-n} \tag{7}$$

where the following notations are accepted

$$\overline{E} = g'(\theta)/g(\theta), \quad \overline{G} = 2\overline{E}^2 - g''(\theta)/g(\theta),$$
$$b_1 = e_1 - e_5 - 2(\lambda - \nu + 1), \quad b_2 = 2e_1 + (\lambda - \nu + 1)e_3,$$
$$b_3 = -e_4, \ e_4 = (\lambda - \mu)e_1 - (\lambda - \mu + 1)e_2, \quad e_5 = (\lambda - \mu) + (\lambda - \mu + 1)\nu$$
$$d_1 = \left[s_1 - 2(\lambda - \mu + 1)s_3 - (\lambda - \mu)(1-\nu) - (\lambda - \mu + 1)(\lambda - \mu)\nu\right]/(1-\nu),$$
$$d_2 = \left[2(\lambda - \mu + 1) - 2s_1\right]/(1-\nu),$$
$$d_3 = (\lambda - \mu)[(\lambda - \mu + 1)s_2 - s_1]/(1-\nu), \quad \mu = (m\lambda + 1)/(n+1)$$
$$\tilde{\sigma}_e = \left(\tilde{\sigma}_{rr}^2 - \tilde{\sigma}_{rr}\tilde{\sigma}_{\theta\theta} + \tilde{\sigma}_{\theta\theta}^2 + 3\tilde{\sigma}_{r\theta}^2\right)^{1/2} \ \text{(for plane stress)}$$
$$\tilde{\sigma}_e = \frac{\sqrt{3}}{2}\left[\left(\tilde{\sigma}_{rr} - \tilde{\sigma}_{\theta\theta}\right)^2 + 4\tilde{\sigma}_{r\theta}^2\right]^{1/2} \ \text{(for plane strain)}$$

with boundary conditions

$$f(0) = 1, \quad f'(0) = 0, \quad f'''(0) = 0, \quad g'(0) = 0, \tag{8}$$
$$f(\pi) = 0, \quad f'(\pi) = 0. \tag{9}$$

Thus, the nonlinear eigenvalue problems (5), (7) and (6), (7) with boundary conditions (8), (9) are formulated. It can be shown [1-4] that it is necessary to modify the traction free boundary conditions. The careful numerical analysis of the problem formulated above shows that it is impossible to find the eigenvalue λ such that the boundary traction-free condition on the crack face is satisfied. It turns out that the function $g(\theta)$ from some value of polar angle θ becomes negative what emphatically it is not. The negative values $g(\theta)$ of contradict to the physical sense of the continuity parameter. To eliminate this difficulty the modified statement of the problem is proposed. Whereupon, a set of favourable boundary conditions are given as

$$f(\theta = \theta_d) = 0, \quad f'(\theta = \theta_d) = 0, \quad g(\theta = \theta_d) = 0 \tag{10}$$

In [1] the two-point boundary value (5), (7) with boundary (10) is studied. A series of numerical results realized in [1] show the dependence of eigenvalue λ on the values of material parameters m, n. In this contribution a series of numerical experiments for plane stress and plane strain conditions are fulfilled. Thorough analysis of the nonlinear eigenvalue problems considered allows to find the analytical presentation of the angular distributions of the stress component and the continuity parameter determined by the analytical solution of (5), (7), (10) and (6), (7), (10)

$$f(\theta) = \frac{\kappa(\cos\theta)^{\mu+2}}{(\lambda+2)(\lambda+1)}, \quad g(\theta) = \kappa^{m/(n+1)}(\cos\theta)^{\mu} \quad 0 \le \theta \le \theta_d \quad \theta_d = \frac{\pi}{2},$$
$$\lambda = \mu = \frac{1}{1+n-m}. \tag{11}$$

The asymptotic fields in the vicinity of the fatigue crack have the form

$$\sigma_{\theta\theta}(r,\theta) = \kappa r^{1/(1+n-m)}(\cos\theta)^{2+\mu}, \quad \sigma_{r\theta}(r,\theta) = \kappa r^{1/(1+n-m)}(\cos\theta)^{1+\mu}\sin\theta,$$
$$\sigma_{rr}(r,\theta) = \kappa r^{1/(1+n-m)}\sin\theta(\cos\theta)^{\mu}, \quad \psi(r,\theta) = \kappa^{m/(n+1)} r^{1/(1+n-m)}(\cos\theta)^{\mu}.$$

The stress and continuity distribution are valid for plane stress and plane strain conditions. The results are presented in Fig.2-4.

Perturbation theory method and its application for solving the nonlinear eigenvalue problems. The exact solution of stresses and continuity is found on the basis of numerical analysis of the boundary value problems for nonlinear ordinary differential equations. This raises the question of whether Eq. (11) can be determined in a general case? It turns out that the perturbation method technique can be easily applied for nonlinear eigenvalue problems and hence it is possible to obtain the eigenvalues of nonlinear eigenvalue problem in the closed form.

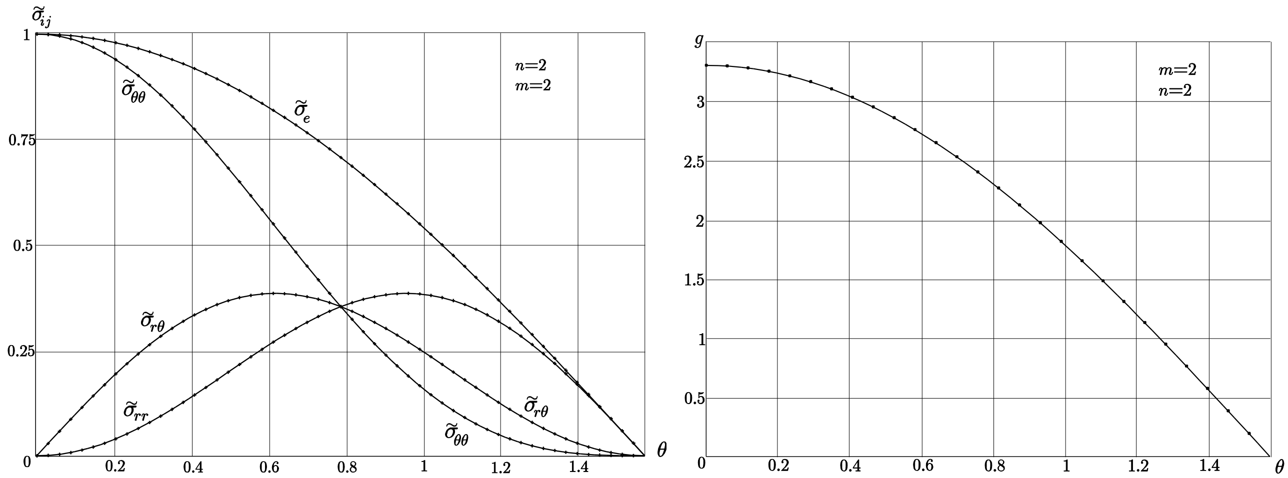

Fig.2. Normalized angular distributions of stress and angular continuity distribution for $n=2, m=2$

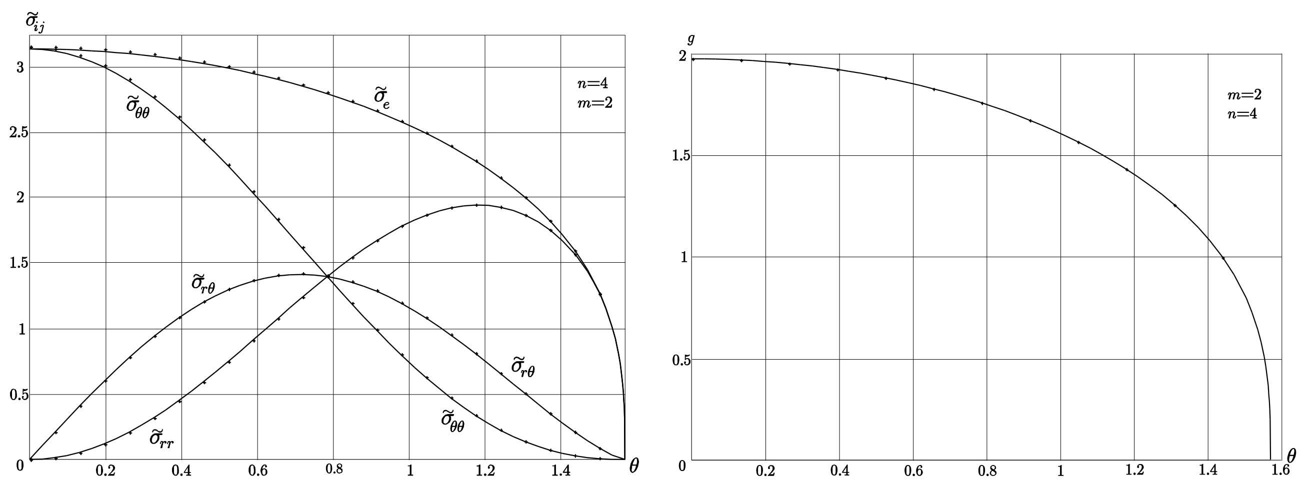

Fig. 3. Normalized angular distributions of stress and angular continuity distribution for $n=4, m=2$

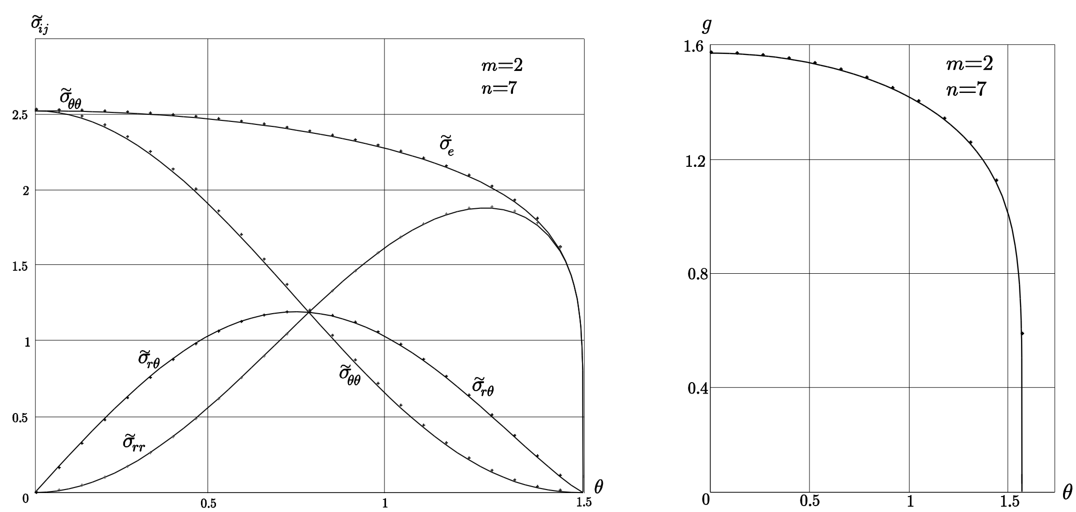

Fig. 4. Normalized angular distributions of stress and angular continuity distribution for $n=7, m=2$

One can introduce a small parameter

$$\varepsilon = \mu - \mu_0 \tag{12}$$

reflecting nonlinearity of the problem: μ_0 is eigenvalue of linear "undisturbed" problem when $n=1, m=1$ are valid; μ is eigenvalue of nonlinear "disturbed" problem when $n>1, m>1$.

We seek a third-order uniform expansion of expanding the angular functions $f(\theta)$, $g(\theta)$ and parameters m, n:

$$\begin{aligned} & f(\theta) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \varepsilon^3 f_3(\theta) + ..., \\ & g(\theta) = g_0(\theta) + \varepsilon g_1(\theta) + \varepsilon^2 g_2(\theta) + \varepsilon^3 g_3(\theta) + ..., \\ & \mu = \mu_0 + \varepsilon , \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \varepsilon^3\lambda_3 + ... \\ & n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + ..., \\ & m = 1 + \varepsilon m_1 + \varepsilon^2 m_2 + \varepsilon^3 m_3 + ... \end{aligned} \tag{13}$$

Introducing asymptotic expansions (12) and (13) into Eqs. (5), (6) and (7) and collecting terms of equal power in the small parameter the set of linear differential equations is obtained.

The zeroth-order problem for plane strain conditions has the form

$$(1-\nu) f_0^{IV} - 2(1-\nu)E_0 f_0^{'''} + (1-\nu)G_0 f_0^{''} + b_1^0 f_0^{''} - 2E_0 b_2^0 f_0^{'} + e_1^0 G_0 f_0 + b_3^0 f_0 = 0 , \tag{14}$$

$$g_0^{'} \sin\theta - \mu_0 \cos\theta = -\sigma_e^{(0)} / g_0 , \tag{15}$$

where the following notations are adopted

$$E_0 = g_0^{'}(\theta) / g_0(\theta) , \quad G_0 = 2E_0^2 - g_0^{''} / g_0 ,$$

$$b_1^0 = 7 - 9\nu, \quad b_2^0 = 5 - 9\nu, \quad b_3^0 = 0, \quad e_1^0 = 3 - 9\nu ,$$

$$\sigma_e^{(0)} = \sqrt{\left[f_0'' - \lambda_0(\lambda_0 + 2) f_0\right] + 4(\lambda_0 + 1)^2 (f_0')^2} .$$

The solution of Eqs. (14), (15) should satisfy the symmetry conditions

$$f_0'(\theta = 0) = 0, \quad f_0'''(\theta = 0) = 0, \quad g_0'(\theta = 0) = 0 , \tag{16}$$

the regularity requirement

$$g_0(\theta = 0) = \left(\sigma_e^{(0)}(\theta = 0)\right)^{1/2} \tag{17}$$

and the traction free condition at $\theta = \theta_d$

$$f_0(\theta = \theta_d) = 0, \quad f_0'(\theta = \theta_d) = 0, \quad g_0(\theta = \theta_d) = 0. \tag{18}$$

The solution of the boundary value problem (14) – (18) can be presented in the form

$$f_0(\theta) = \frac{1}{6}\cos^3\theta, \quad g_0(\theta) = \cos\theta, \quad \mu_0 = 1, \quad \theta_d = \frac{\pi}{2} . \tag{19}$$

The first-order boundary value problem can be formulated as

$$\begin{aligned} & (1-\nu) f_1^{IV} - 2(1-\nu)E_0 f_1^{'''} + (1-\nu)G_0 f_1^{''} + b_1^0 f_1^{''} - 2E_0 b_2^0 f_1^{'} + e_1^0 G_0 f_1 + b_3^0 f_1 - \\ & - \left[(1-\nu) f_0^{''} + e_1^0 f_0\right] g_1^{''} / g_0 + \left[(1-\nu)\left(f_0^{''} g_0^{'} / g_0 - 2 f_0^{'''}\right) / g_0 - 2b_2^0 f_0^{'} / g_0 + 4e_1^0 f_0 g_0^{'} / g_0^2\right] g_1^{'} + \\ & + \left\{(1-\nu)\left[2\frac{f_0^{'''} g_0^{'}}{g_0^2} - 4\frac{f_0^{''} g_0^{'2}}{g_0^3} + \frac{f_0^{''} g_0^{''}}{g_0^2}\right] + \left[2b_2^0 \frac{f_0^{'} g_0^{'}}{g_0^2} - 4e_1^0 \frac{f_0 g_0^{'2}}{g_0^3} + e_1^0 \frac{f_0 g_0^{''}}{g_0^2}\right]\right\} g_1 = \\ & = -b_1^1 f_0^{''} + 2b_2^1 \frac{f_0^{'} g_0^{'}}{g_0} - e_1^1 G_0 - b_3^1 f_0 , \end{aligned} \tag{20}$$

$$
\begin{aligned}
&\left(\frac{f_0^{''}-3f_0}{g_0\sigma_e^{(0)}}\right)f_1^{'''}+\left[\frac{f_0^{'''}-3f_0^{'}}{g_0\sigma_e^{(0)}}+16\frac{f_0^{'}}{g_0\sigma_e^{(0)}}-\frac{f_0^{''}-3f_0}{g_0\left(\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)^{'}\right]f_1^{''}+\\
&+\left[-3\frac{f_0^{'''}-3f_0^{'}}{g_0\sigma_e^{(0)}}-3\frac{f_0^{''}-3f_0}{g_0\sigma_e^{(0)}}+16\frac{f_0^{''}}{g_0\sigma_e^{(0)}}-16\frac{\left(g_0\sigma_e^{(0)}\right)^{'}}{g_0^{\,2}\left(\sigma_e^{(0)}\right)^2}f_0^{'}\right]f_1^{'}+\\
&+\left[-3\frac{f_0^{'''}-3f_0^{'}}{g_0\sigma_e^{(0)}}+3\frac{f_0^{''}-3f_0}{\left(g_0\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)^{'}\right]f_1+\sin\theta g_1^{''}-\\
&-\left(\frac{\sigma_e^{(0)}}{g_0^2}\right)g_1^{'}+\left[\sin\theta-\left(\frac{\sigma_e^{(0)}}{g_0^2}\right)^{'}\right]g_1=\\
&=\frac{\left(f_0^{'''}-3f_0^{'}\right)4\lambda_1 f_0}{g_0\sigma_e^{(0)}}+4\lambda_1\frac{f_0^{''}-3f_0}{g_0\sigma_e^{(0)}}f_0^{'}-16\lambda_1\frac{f_0^{'}f_0^{''}}{g_0\sigma_e^{(0)}}+\\
&+\frac{\left(f_0^{''}-3f_0\right)4\lambda_1 f_0+8\lambda_1\left(f_0^{'}\right)^2}{g_0^{\,2}\left(\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)^{'}+g_0^{'}\cos\theta-g_0\sin\theta-\\
&-\left[\left(\frac{\sigma_e^{(0)}}{g_0}\right)\left(m_1\ln\sigma_e^{(0)}-n_1\ln g_0\right)\right]^{'}.
\end{aligned}
\tag{21}
$$

Eqs. (20), (21) comprise the system of two linear inhomogeneous equations with respect to functions $f_1(\theta)$ and $g_1(\theta)$. Since the homogeneous problem (14)-(18) has a nontrivial solution, the inhomogeneous problem (20), (21) has a solution only if a solvability conditions is satisfied. The solvability condition can be formulated by using a solution of the self-adjoint problem [6]:

$$
\int_0^{\pi/2}\left(H_1\psi_4-H_2\psi_6\right)d\theta=0\,,
\tag{22}
$$

where

$$
\begin{aligned}
H_1=&\left(\begin{aligned}
&\frac{\left(f_0^{'''}-3f_0^{'}\right)4\lambda_1 f_0}{g_0\sigma_e^{(0)}}+4\lambda_1\frac{f_0^{''}-3f_0}{g_0\sigma_e^{(0)}}f_0^{'}-16\lambda_1\frac{f_0^{'}f_0^{''}}{g_0\sigma_e^{(0)}}+\\
&+\frac{\left(f_0^{''}-3f_0\right)4\lambda_1 f_0+8\lambda_1\left(f_0^{'}\right)^2}{g_0^{\,2}\left(\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)^{'}+g_0^{'}\cos\theta-g_0\sin\theta-\\
&-\left[\left(\frac{\sigma_e^{(0)}}{g_0}\right)\left(m_1\ln\sigma_e^{(0)}-n_1\ln g_0\right)\right]
\end{aligned}\right)\frac{1}{(1-\nu)\sin\theta}+\\
&+\left(-b_1^1 f_0^{''}+2b_2^1\frac{f_0^{'}g_0^{'}}{g_0}-e_1^1G_0-b_3^1 f\right)/(1-\nu),\\
H_2=&\left(\begin{aligned}
&\frac{\left(f_0^{'''}-3f_0^{'}\right)4\lambda_1 f_0}{g_0\sigma_e^{(0)}}+4\lambda_1\frac{f_0^{''}-3f_0}{g_0\sigma_e^{(0)}}f_0^{'}-16\lambda_1\frac{f_0^{'}f_0^{''}}{g_0\sigma_e^{(0)}}+\\
&+\frac{\left(f_0^{''}-3f_0\right)4\lambda_1 f_0+8\lambda_1\left(f_0^{'}\right)^2}{g_0^{\,2}\left(\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)^{'}+g_0^{'}\cos\theta-g_0\sin\theta-\\
&-\left[\left(\frac{\sigma_e^{(0)}}{g_0}\right)\left(m_1\ln\sigma_e^{(0)}-n_1\ln g_0\right)\right]
\end{aligned}\right)/\sin\theta,
\end{aligned}
$$

ψ_k are the solution of the adjoint problem corresponding to (20), (21):

$$\psi_1' = a_3\psi_4 + d_3\psi_6$$
$$\psi_2' = -\psi_1 + a_2\psi_4 + d_2\psi_6$$
$$\psi_3' = -\psi_2 + a_1\psi_4 + d_1\psi_6$$
$$\psi_4' = -\psi_3 + a_0\psi_4 + d_0\psi_6$$
$$\psi_5' = a_5\psi_4 + d_5\psi_6$$
$$\psi_6' = -\psi_5 + a_4\psi_4 + d_4\psi_6,$$

where $a_0 = \left(p_1q_5 - p_5q_1\right)/(p_0q_5)$, $a_1 = \left(p_2q_5 - p_5q_2\right)/(p_0q_5)$, $a_2 = \left(p_3q_5 - p_5q_3\right)/(p_0q_5)$,
$a_3 = \left(p_4q_5 - p_5q_4\right)/(p_0q_5)$, $a_4 = \left(p_6q_5 - p_5q_6\right)/(p_0q_5)$, $a_5 = \left(p_7q_5 - p_5q_7\right)/(p_0q_5)$,
$p_0 = 1-\nu$, $p_1 = -2(1-\nu)E_0$, $p_2 = (1-\nu)G_0 + b_3^0$, $p_3 = -2b_2^0E_0$, $p_4 = e_1^0G_0 + b_3^0$,

$$p_5 = -(1-\nu)\frac{f_0''}{g_0} - e_1^0\frac{f_0}{g_0},\ p_6 = (1-\nu)\left[-\frac{f_0'''}{g_0} + 4\frac{f_0''g_0}{g_0^2}\right] - 2b_2^0\frac{f_0'}{g_0} + e_1^0 f_0\frac{4g_0'}{g_0^2},$$

$$p_7 = (1-\nu)\left[\frac{f_0'''g_0'}{g_0^2} - 4\frac{f_0''g_0'^2}{g_0^3} + \frac{f_0''g_0''}{g_0^2}\right] + 2b_2^0\frac{f_0'g_0'}{g_0^2} - 4e_1^0\frac{f_0g_0'}{g_0^3} + e_1^0\frac{f_0g_0''}{g_0^2},$$

$$q_1 = \frac{f_0'' - 3f_0}{g_0\sigma_e^{(0)}},\quad q_2 = \frac{f_0''' - 3f_0'}{g_0\sigma_e^{(0)}} + 16\frac{f_0'}{g_0\sigma_e^{(0)}} - \frac{f_0'' - 3f_0}{g_0^2\left(\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)',$$

$$q_3 = -3\frac{f_0''' - 3f_0'}{g_0\sigma_e^{(0)}} - 3\frac{f_0'' - 3f_0}{g_0\sigma_e^{(0)}} + 16\frac{f_0''}{g_0\sigma_e^{(0)}} - 16\frac{\left(g_0\sigma_e^{(0)}\right)'}{g_0^2\left(\sigma_e^{(0)}\right)^2}f_0',\quad q_4 = -3\frac{f_0''' - 3f_0'}{g_0\sigma_e^{(0)}} + 3\frac{f_0'' - 3f_0}{g_0^2\left(\sigma_e^{(0)}\right)^2}\left(g_0\sigma_e^{(0)}\right)',$$

$$q_5 = \sin\theta,\quad q_6 = -\sigma_e^{(0)}/g_0,\quad q_7 = \sin\theta - \left(\sigma_e^{(0)}/g_0\right)'.$$

The boundary conditions of the adjoint problem are

$$\psi_1(0) = 0,\quad \psi_3(0) = 0,\quad \psi_5(0) = 0,$$
$$\psi_3(\pi/2) = 0,\quad \psi_4(\pi/2) = 0,\quad \psi_6(\pi/2) = 0.$$

The solvability condition (22) enables to obtain the first perturbation of $n-m$: $n_1 - m_1 = -1$. The analytical solution of the boundary value problem (20), (21) can be found

$$f_1(\theta) = \frac{1}{6}\cos^3\theta\left(\ln\cos\theta - \frac{5}{6}\right),\quad g_1(\theta) = \cos\theta\ln\cos\theta.$$

The second-order and the third problems have been either analyzed. The solvability conditions for each of the inhomogeneous system have been considered and studied. The solvability condition for the problem with respect to functions $f_2(\theta)$ and $g_2(\theta)$ allows to find $n_2 - m_2 = 1$. The analytical solution for the angular function $f_2(\theta)$ and $g_2(\theta)$ have the following form

$$f_2(\theta) = \frac{1}{12}\cos^3\theta(\ln\cos\theta)^2 - \frac{5}{36}\cos^3\theta\ln\cos\theta + \frac{19}{216}\cos^3\theta,\quad g_2(\theta) = \cos\theta(\ln\cos\theta)^2.$$

Analysis of the solvability condition for the boundary value problem for the function $f_3(\theta)$ and $g_3(\theta)$ results in the three-term asymptotic expansion of the difference $n-m$:

$$n - m = -\varepsilon + \varepsilon^2 - \varepsilon^3 + O(\varepsilon^4) \qquad (23)$$

Using the asymptotic series of the Poincare type (23) (the straightforward expasion) one can construct the Pade approximation

$$n - m = -\frac{\varepsilon}{1+\varepsilon}.$$

Eliminating by the use of Eq. (12) the small parameter from the equation obtained one can find the final formula

$$n - m = \frac{1-\mu}{\mu} \text{ and } \mu = \frac{1}{1+n-m}.$$

Analysis of the asymptotic expansion for the function $f(\theta)$ and $g(\theta)$ enables to find the exact solution of the nonlinear eigenvalue problem

$$f(\theta) = \frac{\cos^{\mu+2}\theta}{(\lambda+2)(\lambda+1)}, \quad g(\theta) = \cos^{\mu}\theta.$$

Hence, the perturbation theory technique permits to deduce the exact solution of the nonlinear eigenvalue problem arising from nonlinear fracture mechanics analysis and to justify the numerical solutions of singular perturbation problems.

Summary. The new analytical presentation of stress, strain and continuity fields both for plane strain and plane stress conditions is given. The results obtained differ from Zhao and Zhang's solution where the original formulation of the problem for plane stress conditions has been proposed. An analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained. The perturbation technique is used. The method allows to find the analytical formula expressing the eigenvalue as the function of parameters of the damage evolution law.

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