# Construction of Diagrams for Quasi-brittle and Quasi-ductile Fracture of Materials Based on Necessary and Sufficient Criteria

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**Abstract.** Materials characterized by a regular structure and possessing quasi-brittle or quasi-ductile fracture type are investigated, the specific linear size of a structured element being known. When both necessary and sufficient criteria are derived, the Neuber-Novozhilov approach is used. The modified Leonov-Panasyuk-Dugdale the model is proposed for the mode I crack when the pre-fracture zone width coincides with the plasticity zone width. Simple relations for critical parameters of quasi-brittle fracture have been derived: tensile stresses, pre-fracture zone lengths, and stress intensity factors (SIFs). The modified Leonov-Panasyuk-Dugdale model is proposed for the mode I crack when the pre-fracture zone width coincides with the plasticity zone width. Fracture diagrams for which critical stresses calculated trough both criteria are plotted within a wide range of variations in crack lengths. Applying the finite element method, an elastic-plastic problem for tension of a plate with the central crack; sizes and the shape of a plastic zone in the vicinity of the crack tip has been determined at various loading levels corresponding to quasi-brittle and quasi-ductile fracture types. Analysis of the results obtained allows appreciation of the pre-fracture zone width and critical COD (crack opening displacement).

## Introduction

Recently the increasing interest has been viewed just as in multi-scale calculations, so in multi-scale material engineering [1–3]. Previously necessary multi-scale fracture criteria were derived in [4, 5] and in doing so, only initial crack lengths were used. Later in [6–8], the advanced fracture criteria were proposed within the framework of the Leonov-Panasyuk-Dugdale model [9, 10] for one of structural levels. As opposed the classical model, the modification of the Lenov-Panasyuk-Dugdale model resolves properly to the fact that the width of pre-fracture zone in addition to its length was introduced into the model. Introduction of an additional parameter allowed appreciation of fracture of the pre-fracture zone structure nearest to the center of a real crack with invoking information on parameters of the standard $\sigma-\varepsilon$ diagram of material [7]. The derived sufficient criteria [6–8] allow the passage to the limit to necessary criteria when the pre-fracture zone length vanishes.

The Neuber-Novozhilov approach [11, 12] allows one to extend the class of solutions for structured media. According to Novozhilov's terminology, criteria studied here are referred to as sufficient ones. Infinite stresses at the imaginary crack tip, which are excluded by the continual fracture criterion, are not in contradiction with discrete fracture criteria if the singular component of solution has an integrable singularity. The substantiation of the Neuber-Novozhilov approach in formulation of the criteria are given in [13].

## Description of material structure in the proposed model

Materials with a regular structure displaying quasi-brittle or quasi-ductile fracture type are considered, the specific linear size $r_0$ of a structural element (for example, grain size) being known. Make use of the modification [6, 7] of the Leonov-Panasyuk-Dugdale model [9, 10] in which parameters of the classical $\sigma-\varepsilon$ diagram of material strain are used. The simplest approximation of the diagram is shown in Fig 1, *a*. Here $\sigma_m$ is the theoretical (ideal) tensile strength of material, $\varepsilon_0$ is the maximum elastic elongation, and $\varepsilon_1$ is the maximum elongation of a structural element. Thus, the $\varepsilon_1-\varepsilon_0$ value is inelastic elongation of structural element. In the classical Leonov-Panasyuk-Dugdale model, stresses $\sigma_m$ coincide with compression stresses acting on the continuation of a real crack in the pre-fracture zone. Stresses $\sigma_m$ for common structural steels are equal to the yield strength.

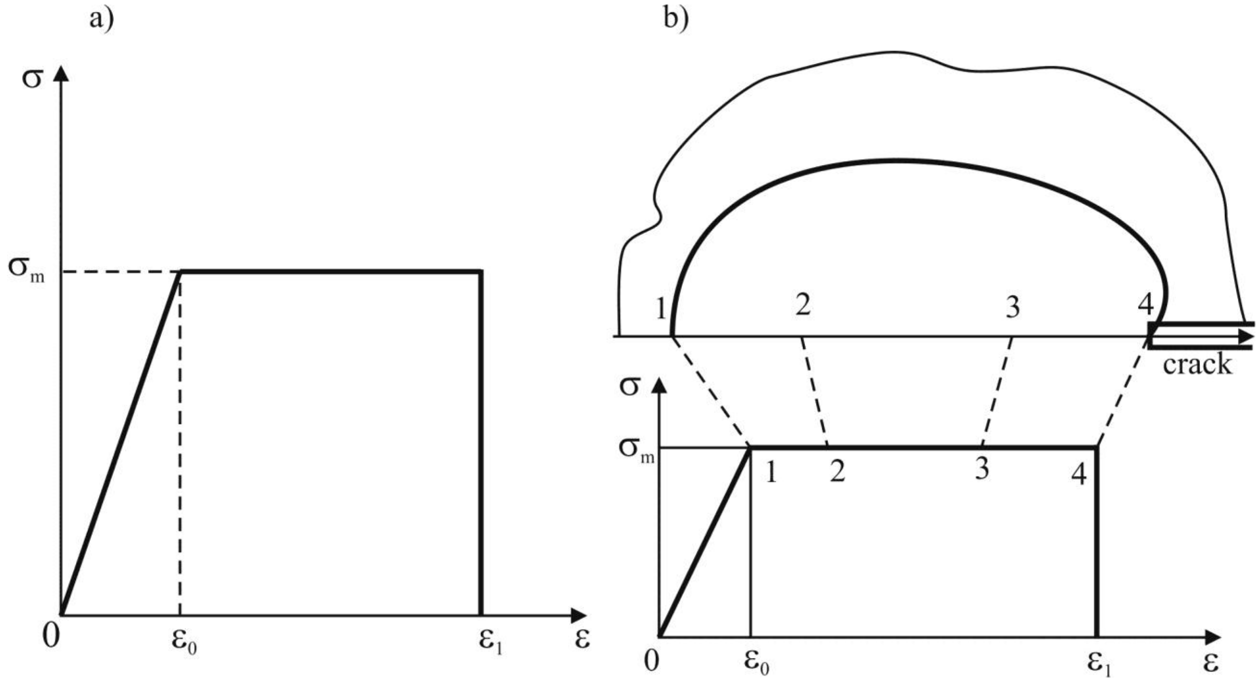


Fig. 1. Approximation of $\sigma-\varepsilon$ material diagram (*a*); interrelation between points $\sigma-\varepsilon$ -of diagram and points of pre-fracture zone (*b*).

Suppose that an internal mode I crack of length $2l_0$ extends rectilinearly in a structurally inhomogeneous material under action of tensile stresses $\sigma_\infty$ specified at infinity. In addition to the real internal rectilinear crack-cut of length $2l_0$, we introduce into consideration an imaginary crack-cut of length $2l=2l_0+2\Delta$, where $\Delta$ is the length of pre-fracture zone located on the continuation of the real crack. In the Leonov-Panasyuk-Dugdale model [9, 10], the field of normal stresses $\sigma_y(x,0)$ on the imaginary crack continuation may be as a sum of two terms (the origin of Cartesian coordinate system $Oxy$ is at the right crack tip of imaginary crack)

$$\sigma_y(x,0)=K_{\mathrm{I}}/(2\pi x)^{1/2}+O(1),\ \ K_{\mathrm{I}}=K_{\mathrm{I}\infty}+K_{\mathrm{I}\Delta},\ \ K_{\mathrm{I}\infty}>0,\ \ K_{\mathrm{I}\Delta}<0, \tag{1}$$

where $K_{\mathrm{I}}=K_{\mathrm{I}}(l,\Delta)>0$ is the total stress intensity factor at the imaginary crack tip; $K_{\mathrm{I}\infty}$ is the stress intensity factor generated by stresses $\sigma_\infty$ specified at infinity; $K_{\mathrm{I}\Delta}$ is the stress intensity factor generated by continued stresses $\sigma_m$ acting in the pre-fracture zone.

The pre-fracture zone [6, 7] occupies a rectangle with sides $\Delta$ and $a$, the pre-fracture zone length $\Delta$ being determined in the process of solving a fracture problem and the pre-fracture zone width $a$ may be identified with the pre-fracture zone width. Given in Fig. 1, *b* is the scheme illustrating qualitatively the interrelation between points 1, 2, 3 and 4 on the $\sigma-\varepsilon$-diagram and those 1, 2, 3 and 4 of the pre-fracture zone. The last points are located on the real crack continuation, its left tip being considered. Beyond the pre-fracture zone, material exhibits elastic deformation and at the pre-fracture zone border, the material begins to exhibit inelastic deformation.

**Discrete-integral fracture criterion**

The sufficient discrete –integral fracture criterion [12] for the mode I crack has the form

$$\frac{1}{r_0}\int_0^{r_0}\sigma_y(x,0)dx \le \sigma_m\,,\ x \ge 0\,; \tag{2}$$

$$2\upsilon(x,0) \le \delta_m^*\,,\ -\Delta \le x < 0\,. \tag{3}$$

Here $\sigma_y(x,0)$ are normal stresses on the crack continuation; $Oxy$ is the Cartesian coordinate system oriented about the right crack tip (the origin of the coordinate system coincides with the imaginary crack tip in the Leonov-Panasyuk-Dugdale model [9, 10]); $r_0$ is the specific linear size of the material structure (grain diameter); $2\upsilon(x,0)$ is the crack opening; $2\upsilon^*(-\Delta^*,0) = \delta_m^*$ is the critical crack opening.

There exists a singular part of solution for a structurally inhomogeneous material, which is determined in relation (1) by the specified stresses $\sigma_\infty$. For every length $2l = 2l_0 + 2\Delta$ of the imaginary crack, stress intensity factor $K_{\mathrm{I}\infty}$ generated by stresses $\sigma_\infty$,has the form $K_{\mathrm{I}\infty} = \sigma_\infty\sqrt{\pi l}$ . In the case of qusi-brittle fracture ($\Delta \ll l_0$), we obtain the following relation with an accuracy of magnitudes of the high order of infinitesimal $K_{\mathrm{I}\infty} \approx \sigma_\infty\sqrt{\pi l_0}$ .

Since $K_{\mathrm{I}} > 0$, we make use of the following representation of normal stresses (1) on the crack continuation

$$\sigma_y(x,0) = \frac{K_{\mathrm{I}\infty}}{\sqrt{2\pi x}} + \sigma_\infty + \frac{K_{\mathrm{I}\Delta}}{\sqrt{2\pi x}}\,. \tag{4}$$

The expression for the stress intensity factor $K_{\mathrm{I}\Delta}$ is written as

$$K_{\mathrm{I}\Delta} = -\sigma_m\sqrt{\pi l}\left[1 - \frac{2}{\pi}\arcsin\left(1 - \frac{\Delta}{l}\right)\right] \approx -\sigma_m\sqrt{\pi l_0}\left[1 - \frac{2}{\pi}\arcsin\left(1 - \frac{\Delta}{l_0}\right)\right]. \tag{5}$$

Thus the distribution of normal stresses on the crack continuation may be given as

$$\sigma_y(x,0) = \sigma_{y\infty}(x,0) + \sigma_{y\Delta}(x,0)\,, \tag{6}$$

where $\sigma_{y\infty}(x,0) = K_{\mathrm{I}\infty}/\sqrt{2\pi x} + \sigma_\infty$, $\sigma_{y\Delta}(x,0) = K_{\mathrm{I}\Delta}/\sqrt{2\pi x}$ .

In papers [6, 7], modification of the Leonov-Panasyuk-Dugdale model has been proposed [9, 10]. Here the parameter characterizing the pre-fracture zone width is introduced into this model and is considered. The scheme illustrating the two-sheet solution is shown in Fig. 2, *a*. The solution defined for the entire plane with a bilateral cut is in agreement with the linear fracture mechanics. One or another solution is defined in compliance with the nonlinear fracture mechanics only for the pre-fracture zone occupying a rectangle with sides $\Delta$ and $a$. Vertexes of this rectangle are $A^+(-\Delta, a/2)$, $B^+(0, a/2)$, $A^-(-\Delta, -a/2)$, $B^-(0, -a/2)$. As it is in the classical model, stresses $\sigma_m$ equal in absolute magnitudes and oppositely directed are applied to crack lips in the pre-fracture zone. This corresponds to the line between points 1 and 4 on the $\sigma - \varepsilon$ diagram depicted in Fig. 1, *b*.

Identify the width of a pre-fracture zone with that of a plasticity zone for the plane stress state at the real crack state [7]: $a = (5/(4\pi))(K_{\mathrm{I}\infty}/\sigma_m)^2$ .We adopt the parameter of the maximum inelastic elongation $\varepsilon_1 - \varepsilon_0$ from the $\sigma - \varepsilon$ diagram (Fig. 1, *a*). Then the critical crack opening $\delta_m$, at which the fiber of a pre-fracture zone nearest to crack center is broken, is calculated by the relation

$$\delta_m^* = (\varepsilon_1 - \varepsilon_0)a \,. \tag{7}$$

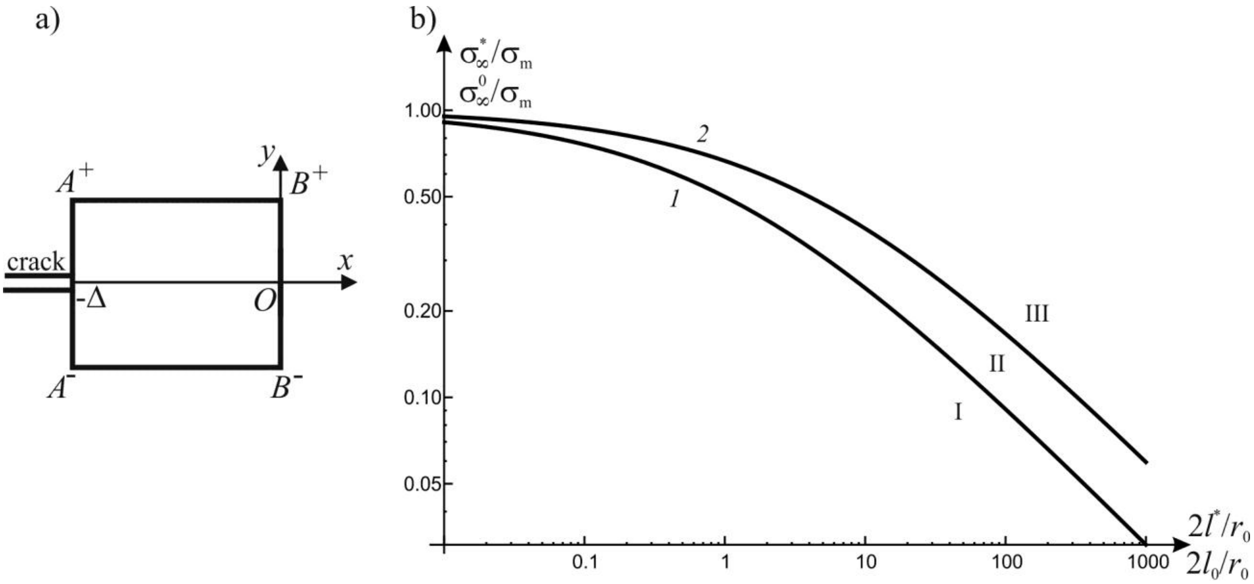


Fig. 2. Arrangement of the rectangular pre-fracture zone (*a*); fracture assessment diagram (*b*).

For opening $2v(x,0)$ for $x<0$ of an imaginary crack in (3), the following representation can be written for the plane stress state

$$2v(x,0) = \frac{\kappa+1}{G} K_{\mathrm{I}} \sqrt{\frac{|x|}{2\pi}} + O(|x|), \quad x<0, \quad K_{\mathrm{I}} > 0\,, \; \kappa = \frac{3-\nu}{1+\nu}\,. \tag{8}$$

Here $\nu$ is the Poisson coefficient and $G$ is the shear modulus.
Transform equalities (2) and (3) making use of relations (4)–(8), retaining terms with multi pliers $\sqrt{\Delta/l}$ and omitting terms with multipliers $\Delta/l \ll 1$. As a consequence, we obtain the analytical expression for critical stresses $\sigma_\infty^*$ for quasi-brittle materials (critical parameters are denoted by asterisks)

$$\frac{\sigma_\infty^*}{\sigma_m} = \left[1 + \left(1 - \frac{5}{8\pi}\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0}\right)\sqrt{\frac{2l^*}{r_0}}\right]^{-1}. \tag{9}$$

Here $2l^* = 2l_0 + 2\Delta^*$ is the critical crack length. Expression (9) makes sense, if

$$1 - \frac{5}{8\pi}\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0} > 0\,. \tag{10}$$

Inequality (10) is the restriction, which holds only for brittle and quasi-brittle materials of the type of ceramics and high-strength alloys. It follows from inequality (10) that the inelastic elongation $\varepsilon_1 - \varepsilon_0$ must not be more than $\approx 5\varepsilon_0$.
For relation (9), the passage to the limit is obvious for $\varepsilon_1/\varepsilon_0 \to 1$ and hence we can turn to consideration of fracture of brittle materials ($\Delta = 0$). Critical stresses $\sigma_\infty^0$ determined by the necessary fracture criterion (2) for brittle fracture are calculated as follows

$$\frac{\sigma_\infty^0}{\sigma_m} = \left(1 + \sqrt{\frac{2l_0}{r_0}}\right)^{-1}. \quad (11)$$

In Fig. 2, *b* are given fracture curves in log-log plots: curve I is plotted by criterion (11), curve *2* is plotted by criterion (9) for the relation $\varepsilon_1 / \varepsilon_0 = 1 + 4\pi / 5$. The fracture assessment diagram, which characterizes the state of nonlinear system on the plane $(2l_0 / r_0, \sigma)$ is plotted. This plane with curves *1* and *2* is divided into three areas: in area I, the initial crack length is not changed (the crack is stable); in area II, the initial crack length increases by the pre-fracture zone length (the crack extends being stable); in area Ш, the initial crack length increases catastrophically (the crack is unstable). The proposed fracture assessment curve agrees well with the experimental results described in [14] in Fig. 1, *a*.

**Numerical experiments**

Consider fracture process of both quasi-brittle and quasi-ductile materials. For quasi-brittle materials, we compare fracture assessment curves obtained previously with results of numerical experiments performed by the method of finite elements. For quasi-ductile materials, we make use of the same sufficient fracture criterion (2) and (3) to fit it for numerical calculation of a stress field around the crack tip and crack opening at the real crack tip.

Make use of the updated Lagrangian approach of solid mechanics equations [15] preferable for simulation of deformation of solids made from elastoplastic material under great strains. Consider a square plate with the central internal crack subjected to axial tension with stresses specified at $\sigma_\infty$ borders. The plate thickness is 0.4 mm, the plate width $w = 100$ mm, the crack of length $2l_0$ varies from 4 mm to 90 mm. In virtue of two symmetry planes, only one fourth of the plate is simulated in the finite element analysis. In the problem domain, a fine uniform mesh with 250000 four-node square elements is generated, each being of 0.1 mm in length. For short cracks of half-lengths $l_0$ = 2, 3, 4 and 5 mm, a fine mesh containing $10^6$ square elements with an element length of 0.05 mm is used. The plate is expected to be deformed under plane stress conditions. The material of the plate has the following characteristics: $E = 200000$ MPa, $\nu = 0.25$ and $\sigma_\mathrm{Y} = 400$ MPa. The fracture diagram of the material is shown in Fig. 1, *a*, where $\sigma_\mathrm{m} = \sigma_\mathrm{Y}$. The external $\sigma_\infty$ load increases following the linear law from zero to $\sigma_\mathrm{Y}$ in a time $t = 1$. By a time in quasi-static problems is meant some monotonically increasing load parameter.

Fig. 3 shows the distribution of equivalent plastic strains $\varepsilon^p = \sqrt{(2/3)\varepsilon_{ij}^p \varepsilon_{ij}^p}$ in the vicinity of the crack tip, which is located on the lower left, for the case $l_0 = 15$ mm, $\sigma_\infty / \sigma_\mathrm{Y} = 0.4$. The narrow plastic zone 2-3 layers of elements thick extends rectilinearly from the crack tip along its axis forming a pre-fracture zone with the width of $\approx 0.1$mm and lengths of $\approx 3.4$ mm, the strain of the elements nearest to the crack tip is $\varepsilon^p = 20\%$. Thus, equality (2) of the strength criterion holds over the whole outline of the plastic zone including the imaginary crack continuation.

For the model of quasi-brittle material, the maximum elongation of a structural element is taken to be $\varepsilon_1 = 0.02$. Then the maximum elastic elongation $\varepsilon_0 = \sigma_\mathrm{T} / E = 0.002$ and inelastic elongation of a structural element $\varepsilon_1 - \varepsilon_0 = 0.018$. Thus we obtain the parameter characterizing the ratio of inelastic and elastic relative material elongations $(\varepsilon_1 - \varepsilon_0) / \varepsilon_0 = 9$. Substituting the pre-fracture zone length $a$ found from numerical calculations into relation (7), we get the critical crack opening $\delta_m^* = (\varepsilon_1 - \varepsilon_0)a$. At the every simulation time-step, the opening of a crack $2\upsilon$ at the point $A^+$ is determined (Fig. 2, *a*) and implementation of the sufficient criterion (3). As soon as $2\upsilon$ becomes larger than $\delta_m^*$ or equal to that, the external load $\sigma_\infty$ registered at this time-step is the critical load determined by the sufficient criterion $\sigma_{\infty}^{*}$. The critical load by the necessary criterion $\sigma_{\infty}^{0}$ is determined at the instant of time corresponding to the first onset of plastic strains in the finite element nearest to the crack tip. Thus equality (3) of the deformation fracture criterion is valid for the real crack tip.

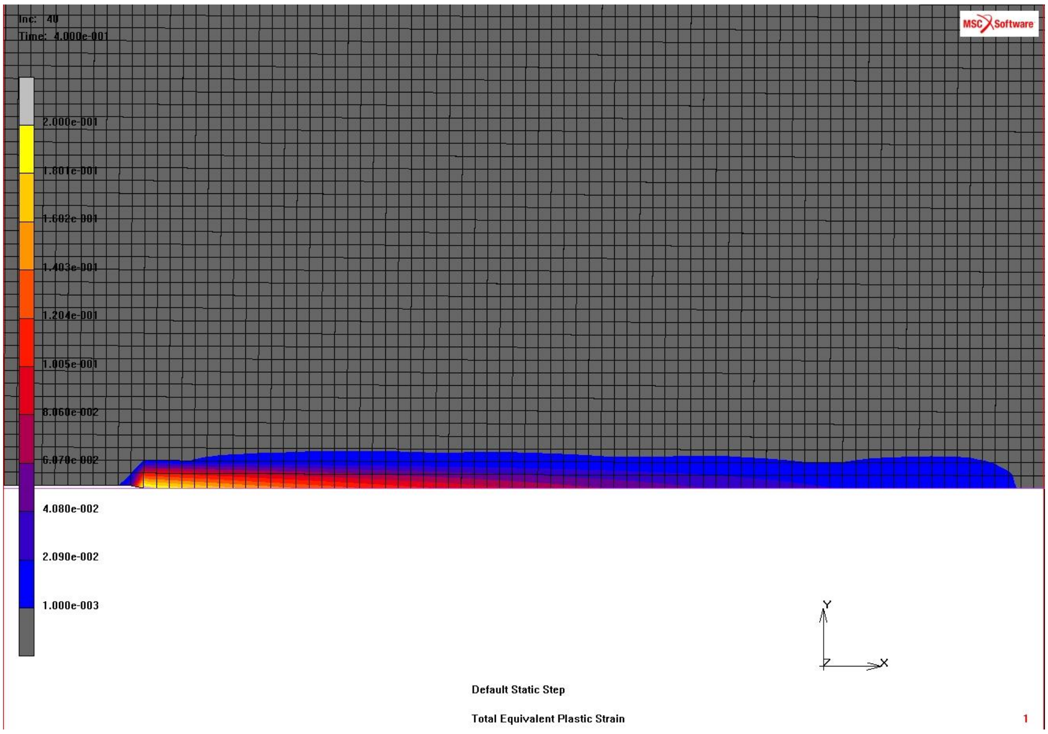


Fig. 3. Zone of plastic strains around the crack tip.

The calculation results are depicted in Fig. 4, *a*, where fracture diagrams are shown in "ordinary" coordinate system: curves *1* and *2* are plotted by the sufficient criterion and the necessary one, respectively; points correspond to numerical calculations for cracks with half-lengths of $l_0 = 2$, 3, 4, 5, 10, 15, …, 45 mm. Approximation of experimental data is carried out by the method of least squares making use of structural relations (11) and (9) for curves *1* and *2*, respectively.

The technique of plotting curve *1*.should be elucidated. On the abscissa in Fig. 2, *b*, the value of $2l_0 / r_0$ is plotted, where $2l_0$ is the initial crack length, $r_0$ is the specific linear size of a material structure, and on the abscissa in Fig. 4, *a*, the value of $l_0$ is plotted. For approximation of numerical calculation data, functions of the kind of $(a+\sqrt{bx})^{-1}$ are chosen, coefficients of which are determined by the method of least squares. This allows the following values to be obtained: $a = 0.9999$ and $b = 16.95$. Equating the coefficient $2/r_0$ at the independent variable $l_0$ in relation (11) to the coefficient $b$, we obtain $2/r_0 = 16.95$, from which $r_0 = 0.118$. The specific linear size of the material structure $r_0$ turns to be 0.1 mm, i.e., it is approximately equal to the size of the finite element.

The parameter of relative inelastic elongation $\left(\varepsilon_1 - \varepsilon_0\right)/\varepsilon_0 = 9$ is appropriate to the quasi-ductile fracture type; therefore, relation (9) is not acceptable since limitation (10) is violated. For approximation of critical stresses by the sufficient criterion, the function of the type $(1+ax^b)^{-1}$ is chosen. The method of least squares gives the value $b = 0.89$ for exponent, which essentially differs from the exponent 0.5 in relation (9). Deviation of the approximating curves *1* and *2* from experimental points for $l_0 > 35$ mm is explained by the effect of finiteness of the plate width since the theory discussed above has been developed for infinite plates. The effect of the plate width on critical fracture curves for edge cracks is given in [16].

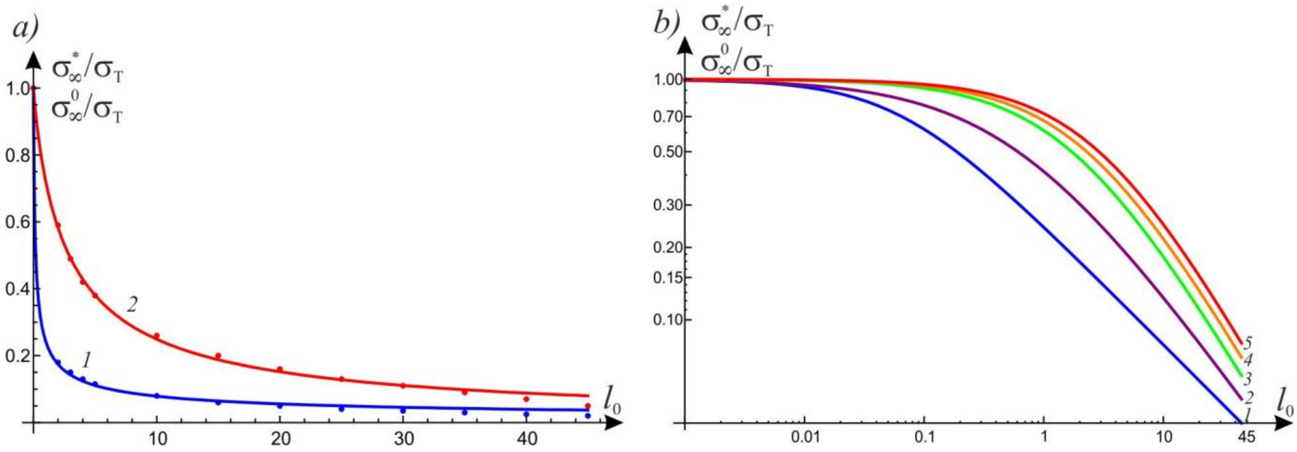


Fig. 4. Fracture diagrams plotted by the method of least squares (*a*); those in the log-log coordinate system (*b*).

Fracture diagrams in Fig. 4, *b* are plotted in log-log coordinates in the wide range of fracture from quasi-brittle to quasi-ductile. Curve *1* is plotted by the necessary strength criterion (11). Curves *2*, *3*, *4* and *5* are plotted by the sufficient criterion for values of the relative inelastic elongation parameter $\left(\varepsilon_1-\varepsilon_0\right)/\varepsilon_0=2$, 5, 7 and 9, respectively. Choice of the approximating function in the form $(1+ax^b)^{-1}$ leads to exponents $b=0.7$, 0.85, 0.87 and 0.89 for curves *2*, *3*, *4* and *5*, respectively. Analysis of numerical results carried out by calculations of $\sigma_\infty^0$ and $\sigma_\infty^*$ and comparison of the calculation results with analytical expressions (9) and (11) makes it possible to infer that analytical representation (11) for brittle fracture gives good agreement with the numerical calculation, and expression (9) can be used for quasi-brittle fracture only for $\left(\varepsilon_1-\varepsilon_0\right)/\varepsilon_0\le 2$.

Given in Fig. 5 is the deformed configuration of a crack surface at 80:1 scale for the case of $l_0=15$ mm, $\sigma_\infty/\sigma_T=0.19$. Blunting of the initially sharp internal crack is observed, therewith the more is the load, the greater is the blunting. In the proposed analytical model (2) and (3), the last condition is appropriate to blunting of the real crack at its tip and estimates break of displacements: an initially sharp crack transforms into blunt one under deformation of elstoplasic material [17]. Earlier Rice and Rosengren [18] and Hutchinson [19] drawn attention to gradual crack blunting when the problem on tension of a plate with the slit was studied for strengthened materials.

**Summary**

Materials possessing brittle, quasi-brittle, and quasi-ductile fracture types have been considered. For these materials, fracture diagrams have been plotted, which contain two critical curves on the plane “crack length – external load”. The lower bound for critical stresses corresponds to the necessary fracture criterion, and the upper bound for critical stresses corresponds to the sufficient criterion. For description of brittle and quasi-brittle fracture, analytical expressions have been derived. Numerical experiment on simulation of quasi-brittle and quasi-ductile fractures in the context of the theory of great elastoplastic strains allowed one to refine the applicability domain of analytical representations for quasi-brittle fracture and to obtain the simple description of a passage from quasi-brittle to quasi-ductile fracture. The results of full-scale experiments [14, 16] on fracture of brittle and quasi-brittle materials agree well with calculation results carried out with a help of analytical expressions and numerical experiment.

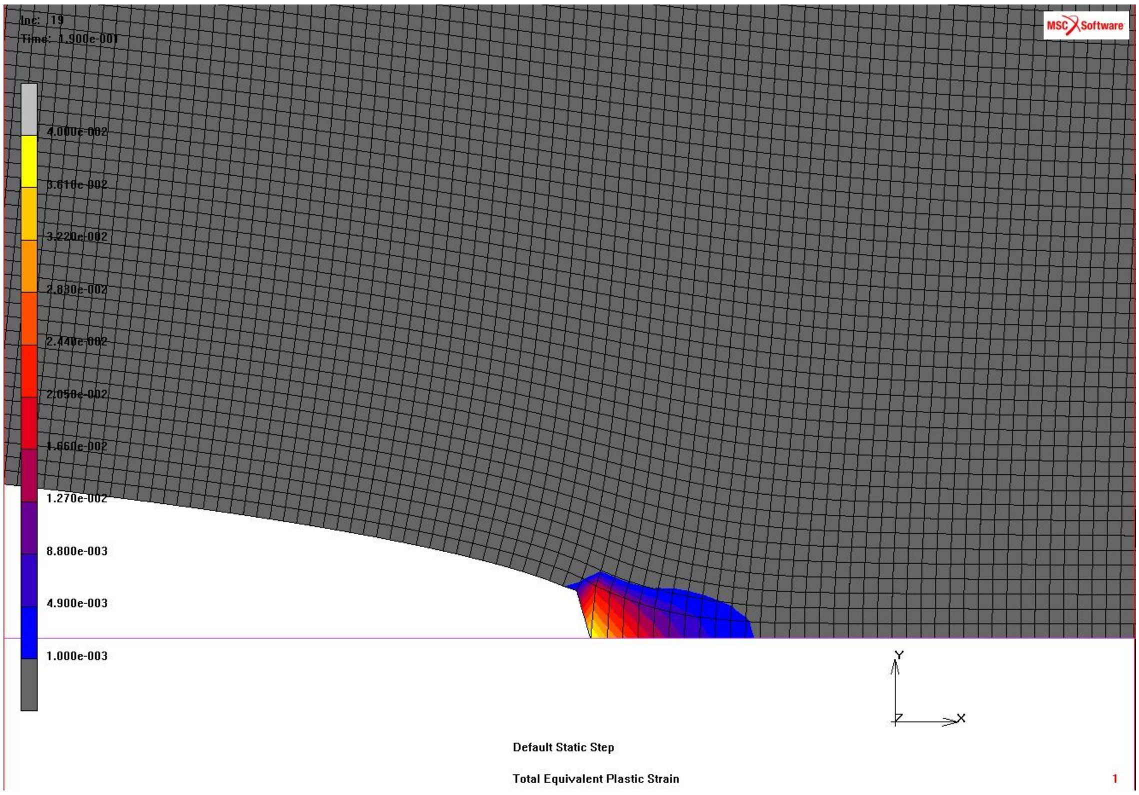


Fig. 5. Deformed configuration of crack surface.

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